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EXTENDED FIXED POINT THEOREMS IN PARTIALLY ORDERED B-METRIC SPACES

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ABSTRACT

In this paper, we enlarge the class of b-metric spaces by considering the class of extended partially ordered b-metric spaces and utilize the same to prove our fixed point results. Our result extends and improves many results of the existing literature. We, also furnish examples to support the genuineness of our results.

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INTRODUCTION

The fixed point theory is very important concept in mathematics. In 1922, Banach [1] proved his classical contraction principle in the concept of fixed point theory. The investigation of existence and uniqueness of fixed point for a self mapping and common fixed points for two or more mappings has become a very active subject of interest. Later many researchers proved Banach contraction principle in multitude of generalized metric spaces. The concept of b-metric space introduced by S. Czerwik [14, 20], which is the generalization of usual metric space and generalized the Banach contraction principal in contrast of complete b-metric space. Ran and Reurings [21] give the concept of partially ordered metric spaces and some of its applications to matrix equation. After that, many researchers [6, 8, 11, 12, 16, 18, 19, 22, 23] carried out further studies on b-metric spaces and partially ordered metric spaces. The aim of this paper is to establish some fixed point theorems for a mapping having a monotone property satisfying a contractive condition of rational type in the partially ordered b-metric spaces. We, also furnish examples to establish the genuineness of our results.

2 Preliminaries

The following definitions and results will be needed in what follows.

Definition 2.1 [20] Let X be a non empty set and $t \geq 1$ be any real number. Let $d : X \times X \to R^+$ be a function satisfying the following conditions :

(a) $d(x, y) = 0$ iff $x = y$,

(b) $d(x, y) = d(y, x)$,

(c) $d(x, y) \leq t[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then the function d is called a b-metric on X and the triplet (X, d,t) is called a b-metric space.

Definition 2.2 [20] Let (X, d) be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent sequence in (X, d) and converges to x, if for every $\epsilon > 0$ there exists $n_0 \in N$ such that $d(x_n, x) < \epsilon$ for all $n > n_0$ and represented by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\epsilon > 0$ there exists $n_0 \in N$ such that $d(x_n, x_{n+p}) < \epsilon$ for all $n > n_0$, $p > 0$ or equivalently $\lim_{n \to \infty} d(x_n, x_{n+p} = 0$ for all $p > 0$.

(c) X is said to be complete if every Cauchy sequence in X converges to $x \in X$.

3 Main results

Theorem 3.1 Let (X, d, ≤) be a complete partially ordered b-metric space and $f : X \to X$ be a continuous non decreasing mapping. Suppose there exist another mapping $c_i : X \times X \to [0, 1)$ such that

$c_i(fp, fq) \leq c_i(p, q)$ for all $p, q \in X$ and i = 1,2,3.....8.

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Also, for all p, q ϵ X with,

$d(fp, fq) \le c_1(p, q)d(p, q) + c_2(p, q)[d(p, fp) + d(q, fq)] + c_3(p, q)\frac{d(q, fp) + d(p, fq)}{t} + c_4(p, q)d(q, fq)g(d(p, q), d(p, fp)) + c_5(p, q)d(q, fp)g(d(p, q), d(p, fq)) + c_6(p, q)d(p, q)\ g(d(p, q), d(p, fp) + d(q, fp)) + c_7(p, q)d(q, fp) + c_8(p, q)d(p, fq).$ (3.1)

where $g : R^+ \times R^+ \to R^+$ is a function such that g(m, m) = 1, for all m ϵ R^+ and

$\text{Sup}_{x, y \in X}[c_1(p, q) + c_2(p, q) + c_3(p, q) + c_4(p, q) + c_6(p, q) + c_8(p, q)] < \frac{1}{t+1}$.

If there exists point $p_0 \in X$ with $p_0 \le fp_0$, then f possesses a fixed point.

Proof : Let $p_0 = fp_0$ then the result is true. Suppose now $p_0 < fp_0$. Define sequence $\{p_n\}$ in X by

$p_{n+1} = fp_n$ for all n = 0, 1, 2,..... (3.2)

Since f is non decreasing mapping so by induction

$p_0 \le fp_0 = p_1 \le fp_1 = p_2 \le \ldots\ldots \le fp_{n-1} = p_n \le fp_n = p_{n+1} \le \ldots\ldots\ldots$ (3.3)

Suppose for some m ϵ N, $p_{m+1} = p_m$, then $p_{m+1} = fp_m = p_m$, then p_m is a fixed point and proof is finished.

Now since $p_n < p_{n+1}$ for all n ϵ N, by (3.1),

$d(p_{n+1}, p_{n+2}) = d(fp_n, fp_{n+1}) \le c_1(p_n, p_{n+1})d(p_n, p_{n+1}) + c_2(p_n, p_{n+1})[d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2})] + c_3(p_n, p_{n+1})\frac{d(p_{n+1}, p_{n+1}) + d(p_n, p_{n+2})}{t} + c_4(p_n, p_{n+1})d(p_{n+1}, p_{n+2})g(d(p_n, p_{n+1}), d(p_n, p_{n+1}))) + c_5(p_n, p_{n+1})d(p_n, p_{n+1})g(d(p_n, p_{n+1}), d(p_n, p_{n+2}))) + c_6(p_n, p_{n+1})d(p_n, p_{n+1})g(d(p_n, p_{n+1}), d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+1}))) + c_7(p_n, p_{n+1})d(p_{n+1}, p_{n+1}) + c_8(p_n, p_{n+1})d(p_n, p_{n+2})$.

$\le c_1(p_n, p_{n+1})d(p_n, p_{n+1}) + c_2(p_n, p_{n+1})[d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2})] + c_3(p_n, p_{n+1})[d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2})] + c_4(p_n, p_{n+1})d(p_{n+1}, p_{n+2}) + c_6(p_n, p_{n+1})d(p_n, p_{n+1}) + c_8(p_n, p_{n+1})[d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2})]$.

$= [c_1(p_n, p_{n+1}) + c_2(p_n, p_{n+1}) + c_3(p_n, p_{n+1}) + c_6(p_n, p_{n+1}) + c_8(p_n, p_{n+1})]\, d(p_n, p_{n+1}) + [c_2(p_n, p_{n+1}) + c_3(p_n, p_{n+1}) + c_4(p_n, p_{n+1}) + c_8(p_n, p_{n+1})]d(p_{n+1}, p_{n+2})$

$= [c_1 + c_2 + c_3 + c_6 + c_8](fp_{n-1}, fp_n)d(p_n, p_{n+1}) + [c_{2+} c_3 + c_4 + c_8](fp_{n-1}, fp_n)d(p_{n+1}, p_{n+2})$

$\le [c_1 + c_2 + c_3 + c_6 + c_8](p_{n-1}, p_n)d(p_n, p_{n+1}) + [c_{2+} c_3 + c_4 + c_8](p_{n-1}, p_n)d(p_{n+1}, p_{n+2})$

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$\le [c_1 + c_2 + c_3 + c_6 + c_8](p_0, p_1)d(p_n, p_{n+1}) + [c_{2+} c_3 + c_4 + c_8](p_0, p_1)d(p_{n+1}, p_{n+2})$

or $d(p_{n+1}, p_{n+2}) \le \frac{(c_1 + c_2 + c_3 + c_6 + c_8)(p_0, p_1)}{1 - (c_2 + c_3 + c_4 + c_8)(p_0, p_1)} d(p_n, p_{n+1})\ d(p_n, p_{n+1})$

$d(p_{n+1}, p_{n+2}) \le \mu\, d(p_n, p_{n+1})$

where $0 < \mu < 1$, since

$\text{Sup}_{x, y \in X}[c_1(p, q) + c_2(p, q) + c_3(p, q) + c_4(p, q) + c_6(p, q) + c_8(p, q)] < \frac{1}{t+1}$.

Continuing this process, we have $d(p_{n+1}, p_{n+2}) \le \mu^{n+1}\, d(p_0, p_1)$

Thus for all m > n, where m, n ϵ N, we have,

$d(p_n, p_m) \le td(p_n, p_{n+1}) + t^2 d(p_{n+1}, p_{n+2}) + \ldots\ldots + t^{m-n} d(p_{m-1}, p_m)$

$\le [t\mu^n + t^2\mu^{n+1} + \ldots\ldots + t^{m-n}\mu^{m-1}]d(p_0, p_1)$

$= t\mu^n(1 + t\mu + (t\mu)^2 + \ldots\ldots + (t\mu)^{m-n-1}]d(p_0, p_1)$

$\le t\mu^n(1 + t\mu + (t\mu)^2 + \ldots\ldots]d(p_0, p_1)$

$\le \frac{t\mu^n}{1 - t\mu} d(p_0, p_1)$

Thus $d(p_n, p_m) \to 0$ as $n \to \infty$ for all m ϵ N. Therefore $\{p_n\}$ is a Cauchy sequence. But X is complete b-metric space so there exist p ϵ X such that

$\text{Lim}_{n \to \infty} p_{n+1} = p$.

Again letting $n \to \infty$ in equation (3.2), we have

$p = \lim_{n \to \infty} p_{n+1} = \lim_{n \to \infty} f(p_n) = f(\lim_{n \to \infty} p_n) = f(p)$.

Hence p is a fixed point of X.

Now we give an example in support of theorem (3.1).

Example 3.2 Consider a metric $d(p, q) = |p - q|^2$ with usual order ≤ and X = [0, 1]. Then obviously d is a b-metric space when s = 2. Define a mapping $c_i : X \times X \to [0, 1)$ such that

$c(p, q) = \frac{p + q}{16}$ and $f_p = \frac{p^2}{8}$ then,

$c(f_p, f_q) = \frac{1}{16}\left(\frac{p^2}{8} + \frac{p^2}{8}\right) \le \frac{p + q}{16} = c(p, q)$

and $d(f_p, f_q) = |fp - fq|^2 = \left|\frac{p^2}{8} - \frac{q^2}{8}\right|^2 = \frac{1}{8^2}|p + q|^2\, |p - q|^2$

$\le \frac{1}{8 \times 4}|p + q|\ \ |p - q|^2$

$\le \frac{1}{4}\ c(p, q)\, d(p, q) = c(p, q)\, d(p, q)$

Also f is non decreasing continuous mapping with usual order ≤. Hence f has a fixed point p = 0.

Corollary 3.3 Let (X, d, ≤) be a complete partially ordered b-metric space and f : X→ X be a continuous non decreasing mapping such that the following conditions hold,

$d(fp, fq) \le c_1\, d(p, q) + c_2[d(p, fp) + d(q, fq)] + c_3\frac{d(q, fp) + d(p, fq)}{t} + c_4\, d(q, fq)g(d(p, q), d(p, fp)) + c_5$

$d(q, fp)g(d(p, q), d(p, fq)) + c_6\, d(p, q)g(d(p, q), d(p, fp) + d(q, fp)) + c_7\, d(q, fp) + c_8\, d(p, fq)$

for all p, q ϵ X with p ≤ q and c_i are non negative coefficients for i = 1, 2,.....,8 with $c_1 + c_2 + c_3 + c_4 + c_6 + c_8 < \frac{1}{t+1}$ and g : $R^+ \times R^+ \to R^+$ is a function such that g(m, m) = 1, for all m ϵ R^+. If there exists point $p_0 \in X$ with $p_0 \le fp_0$, then f possesses a fixed point.

Now we give an example in support of corollary (3.3).

Example 3.4 Consider a metric $d(p, q) = |p-q|^2$ with usual order ≤ and X = [0, 1]. Then obviously d is a b-metric space when s = 2. Define $f_p = \frac{p}{4} - \frac{p^2}{8}$ then, for all p, q ϵ X, we have

$$d(f_p, f_q) = |fp - fq|^2 = \left|\frac{p}{4} - \frac{p^2}{8} - \frac{q}{4} + \frac{q^2}{8}\right|^2 = \left|\frac{1}{4}(p-q) - \frac{1}{8}(p-q)(p+q)\right|^2$$

$$= |p-q|^2 \left|\frac{1}{4} - \frac{1}{8}(p+q)\right|^2$$

$$\le \frac{1}{4} |p-q|^2 = cd(p, q).$$

Also f is non decreasing continuous mapping with usual order ≤ . Hence f has a fixed point p = 0.

Theorem 3.5 Let (X, d, ≤) be a complete partially ordered b-metric space and f : X→ X be a continuous non decreasing mapping. Suppose there exist another mapping c_i : X × X→ [0, 1) such that

$c_i(fp, fq) \le c_i(p, q)$ for all p, q ϵ X and i = 1,2,3.....8.

Also, for all p, q ϵ X with p ≤ q,

$$d(fp, fq) \le c_1(p, q)d(p, q) + c_2(p, q)[d(p, fp) + d(q, fq)] + c_3(p, q)\frac{d(q, fp) + d(p, fq)}{t} + c_4(p, q)\frac{d(q, fq)[1 + d(p, fp)}{1 + d(p, q)} + c_5(p, q)\frac{d(q, fp)[1 + d(p, fq)]}{1 + d(p, q)} + c_6(p, q)\frac{d(p, q)[1 + d(p, fp) + d(q, fp)]}{1 + d(p, q)} + c_7(p, q)d(q, fp) + c_8(p, q)d(p, fq).$$

And $\text{Sup}_{x, y \in X}[c_1(p, q) + c_2(p, q) + c_3(p, q) + c_4(p, q) + c_6(p, q) + c_8(p, q)] < \frac{1}{t+1}$.

If there exists point $p_0 \in X$ with $p_0 \le fp_0$, then f possesses a fixed point.

The proof of this theorem is same as theorem (3.1).

Corollary 3.6 Let (X, d, ≤) be a complete partially ordered b-metric space and f : X→ X be a continuous non decreasing mapping such that the following conditions hold,

$$d(fp, fq) \le c_1 d(p, q) + c_2[d(p, fp) + d(q, fq)] + c_3\frac{d(q, fp) + d(p, fq)}{t} + c_4\frac{d(q, fq)[1 + d(p, fp)}{1 + d(p, q)} + c_5\frac{d(q, fp)[1 + d(p, fq)]}{1 + d(p, q)} + c_6\frac{d(p, q)[1 + d(p, fp) + d(q, fp)]}{1 + d(p, q)} + c_7 d(q, fp) + c_8 d(p, fq).$$

for all p, q ϵ X with p ≤ q and c_i are non negative coefficients for i = 1, 2,.....,8 with $c_1 + c_2 + c_3 + c_4 + c_6 + c_8 < \frac{1}{t+1}$. If there exists point $p_0 \in X$ with $p_0 \le fp_0$, then f possesses a fixed point.

References

1. S. Banach, "Sur les operastions dans les ensembles abstraits et leur application aux equations integrals," Fundamenta Mathematicae, vol. 3(1922), pp. 133-181.
2. A. Aghajani, M. Abbas and J. R. Roshan, A common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Mathematica Slovaka, 64, 2004, 941-960.
3. A. Aghajani and R. Arab, Fixed points of (Ψ, Φ, θ)-contractive mappings in partially ordered b-metric spaces and application to quadratic integral equations, Fixed point theory and applications, 2013, Article I. D. 245.
4. S. Chandok, T. D. Narang and M. A. Taoudi, Some coupled fixed point theorems for mappings satisfying a generalized contractive condition of rational type, Palestine Journal of Mathematics, 4(2), 2015, 360-366.
5. R. Allahyari, R. Arab and A. Shole Haghighi, Fixed points of admissible almost contractive type mappings on b-metric spaces with an application to quadratic integral equations, Journal of inequalities and applications, 2015(2015), Article I. D. 32.
6. M. Boriceanu, M. Bota and A. Petrusel, Multvalued fractals in b-metric spaces, Cent. Eur. J. Math., (8)2010, 367-377.
7. H. Faraji, D. Savi and S. Radenovi, Fixed point Theorems for Geraghty contraction type mappings in b-metric spaces and applications, Axioms, 8(2019), 34.
8. M. Pacurar, Sequences of almost contractions and fixed points in b-metric spaces, Anal. Uni. De Vest, Timisoara Seria Matematica Informatica, 48(2010), 125-137.
9. M. A. Kutbi, E. Karapnar, J. Ahmed, A. Azem, Some fixed point results for multivalued mappings in b-metric spaces, J. Inequal. Appl., 2014(2014), Article I D 126.
10. T. G. Bhasker and V. Laxmikantham, Fixed point theory in partially ordered metric spaces and applications, Nonlinear Anal., 65(2006), 1379-1393.
11. S. Chandok, T. D. Narang and M. Taoudi, Fixed point theorem for generalized contractions satisfying rational type expressions in partially ordered metric spaces, Gulf Journal of Mathematics, 2(2014), 87-93.
12. N. S. Rao and K. Kalyani, Coupled fixed point theorems with rational expressions in partially ordered metric spaces, J. Anal., 28(2020), 1085-1095.
13. W. Shatanawi and A. Pitea, Omega distance and coupled fixed point in G-metric spaces, Fixed point theory and applications, 2013, 208.
14. S. Czerwik, Contraction mappings in b-metric spaces, Acta mathematica et Informatica Universitatis Ostraviensis , 1(1993), 5-11.
15. W. Shatanawi and M. Postolache, Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces, Fixed point theory and applications, 2013, 60.
16. M. A. Kutbi, E. Karapnar, J. Ahmed and A. Azem, Some fixed point results for malt valued mappings in b-metric spaces, J. Inequal. Appl., 2014, Article ID 126.
17. R. P. Agarwal, M. A. El-Gebeily and D. O. Regan, Generalized contractions in partially ordered metric spaces,

Applicable Analysis, 87(2008), 18.

18. R. Arab and k. Zare, Fixed point results for rational types contractions in partially ordered b-metric spaces, Int. Jour. Anl. Appl., 10, 2016, 64-70.
19. R. Arab and k. Zare, Fixed point results for rational types contractions in partially ordered b-metric spaces, Int. Jour. Anl. Appl., 10, 2016, 64-70.
20. S. Czerwik, Non linear set-valued Contraction mappings in b-metric spaces, Atti. Sem. Mat. Fis. Univ. Modena, 46(1998), 263-276.
21. A. C. M. Ran and M. C. B. Reurings, A fixed point theorem in partially ordered sets and some applications to matrix equations, Proceedings of the American Mathematical Society, 132(2004), 1435-1443.
22. R. Allahyari, R. Arab and A. Shole Haghighi, A generalization on weak contraction in partially ordered b-metric spaces and its application to quadratic integral equations, Journal of inequalities and applications, 2014 (2014), Article I. D. 355.
23. T. Abdeljawad, D. Turkoglu, Locally convex valued rectangular metric spaces and Kannan's fixed point theorem, arXiv, 2011(2011), 11 pages. 1.
24. M. Boriceanu, Strict fixed point theorems for multivalued operators in b-metric spaces, Inter. J. Mod. Math., 4(2009), 285-301. 1

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