International Journal of Current Advanced Research
ISSN: O: 2319-6475, ISSN: P: 2319-6505, Impact Factor: 6.614
Available Online at www.journalijcar.org
Volume 7; Issue 5(B); May 2018; Page No. 12309-12314
DOI: http://dx.doi.org/10.24327/ijcar.2018.12314.2159

Research Article

EXISTENCE THE LOCALLY ATTRACTIVE SOLUTION FOR A SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION IN BANACH SPACE

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A R T I C L E I N F O

Article History:

Received 6th February, 2018
Received in revised form 20th March, 2018 Accepted 8th April, 2018
Published online 28th May, 2018

Key words:

Banach space, second order nonlinear differential equation, existence results, locally attractive solution.

A B S T R A C T

In this paper, we prove the existence the locally attractive solution and existence the extremal solution for a second order nonlinear differential equation in Banach space under lipschitz and Caratheodory conditions via a hybrid fixed point theorem.

INTRODUCTION

In this paper the work deals with the existence and locally attractivity of solutions to the following second order Nonlinear Differential equation (SNDE).

$$\left.\begin{array}{c}\mathfrak{D}^2\left[\frac{x(t)}{f\left(t,x(t),x\left(\gamma(t)\right)\right)}\right]=g\left(t,x(t),x\left(\mu(t)\right)\right),t\in\mathbb{R}_+\\ x(0)=0\end{array}\right\} \quad (1)$$

Where, $f:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}-\{0\}$, $g:\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ And $\gamma,\mu:\mathbb{R}_+\to\mathbb{R}$

We use hybrid fixed point theory formulated by B. C. Dhage for the existence of solution of the SNDE (1) and we prove that all the solutions are locally attractive.

Finally we present an example illustrating the applicability of the imposed conditions

By a solution of SNDE (1) we mean a function $x\in\mathcal{AC}(\mathbb{R}_+,\mathbb{R})$ such that:

1. The function $t\to\left[\frac{x(t)}{f(t,x(t),x[\gamma(t)])}\right]$ is absolutely continuous for each $x\in\mathbb{R}$.
2. x satisfies (1)

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Auxiliary Results

In this section we give the definitions, notation, hypothesis and preliminary tools, which will be used in the sequel.

Let $\mathbb{X}=\mathcal{AC}(\mathbb{R}_+,\mathbb{R})$ be the space of absolutely continuous function on $\mathbb{R}_+$ and Ω be a subset of $\mathbb{X}$. Let a mapping $\mathbb{A}:\mathbb{X}\to\mathbb{X}$ be an operator and consider the following operator equation in $\mathbb{X}$ namely, $x(t)=(\mathbb{A}x)(t)$, for all $t\in\mathbb{R}_+$ (1)

Below we give some different characterization of the solutions for operator equation (2.1) on $\mathbb{R}_+$. We need the following definitions.

Definition: We say that solution of the equation (2.1) are locally attractive if there exists a closed ball $\overline{B_r(0)}$ in the space $\mathcal{AC}(\mathbb{R}_+,\mathbb{R})$for some $x_0\in\mathcal{AC}(\mathbb{R}_+,\mathbb{R})$and for some real number $r>0$ such that for arbitrary solution $x=x(t)$ and $y=y(t)$ of equation (2.1) belonging to $\overline{B_r(0)}\cap\Omega$ we have that $\lim_{t\to\infty}\left(x(t)-y(t)\right)=0(2)$

Definition: Let $\mathbb{X}$ be a Banach space. A mapping $\mathbb{A}:\mathbb{X}\to\mathbb{X}$ is called Lipschitz if there is a constant $\alpha>0$ such that,$\|\mathbb{A}x-\mathbb{A}y\|\le\alpha\|x-y\|$for all $x,y\in\mathbb{X}$. If $\alpha<1$, then $\mathbb{A}$ is called a contraction on $\mathbb{X}$ with the contraction constant α.

Definition: An operator $\mathbb{Q}$ on a Banach space $\mathbb{X}$ into itself is called compact if for any bounded subset S of $\mathbb{X}$, $\mathbb{Q}(S)$ is relatively compact subset of $\mathbb{X}$. If $\mathbb{Q}$ is continuous and compact, then it is called completely continuous on $\mathbb{X}$.

Definition: Let $\mathbb{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathbb{Q}: \mathbb{X} \to \mathbb{X}$, be an operator (in general nonlinear). Then $\mathbb{Q}$ is called

1. Compact if $\mathbb{Q}(X)$ is relatively compact subset of $\mathbb{X}$.
2. Totally compact if $\mathbb{Q}(S)$ is totally bounded subset of $\mathbb{X}$ for any bounded subset S of $\mathbb{X}$.
3. Completely continuous if it is continuous and totally bounded operator on $\mathbb{X}$.

It is clear that every compact operator is totally bounded but the converse need not be true.

We seek the solution of (1) in the space $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$of continuous and real – valued function defined on $\mathbb{R}_+$. Define a standard norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ by $\|x\| = sup\{|x(t)|: t \in \mathbb{R}_+\}, \ (xy)(t) = x(t)y(t), \ t \in \mathbb{R}_+$ (3)

Clearly $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$becomes a Banach space with respect to the above norm and the multiplication in it. By $\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ we denote the space of Lebesgue-integrable function $\mathbb{R}_+$with the norm $\|\cdot\|_{\mathcal{L}^1}$ defined by $\|x\|_{\mathcal{L}} = \int_0^\infty |x(t)|dt$ (4)

Definition: Let $f \in \mathcal{L}^1[0, \mathbb{T}]$and $\alpha > 0$. The Riemann – Liouville fractional derivative of order ξ of real function f is defined as $\mathfrak{D}^\xi f(t) = \frac{1}{\Gamma(1-\xi)} \frac{d}{dt} \int_0^t \frac{f(s)}{(t-s)^\xi} ds \quad , \quad 0 < \xi < 1$
Such that $\mathfrak{D}^{-\xi} f(t) = I^\xi f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds$ respectively.

Definition: The Riemann-Liouville fractional integral of order $\xi \in (0,1)$ of the function $f \in \mathcal{L}^1[0, \mathbb{T}]$ is defined by the formula: $I^\xi f(t) = \frac{1}{\Gamma(\xi)} \int_0^t \frac{f(s)}{(t-s)^{1-\xi}} ds, \quad t \in [0, \mathbb{T}]$

where $\Gamma(\xi)$ denote the Euler gamma function. The Riemann-Liouville fractional derivative operator of order ξ defined by $\mathfrak{D}^\xi = \frac{d^\xi}{dt^\xi} = \frac{d}{dt} \circ I^{1-\xi}$.It may be shown that the fractional integral operator I^ξ transforms the space $\mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ into itself and has some other properties.

Theorem: (Arzela-Ascoli Theorem) If every uniformly bounded and equicontinuous sequence $\{f_n\}$ of functions in$\mathcal{C}(\mathbb{R}_+, \mathbb{R})$, then it has a convergent subsequence.

Theorem: A metric space X is compact iff every sequence in X has a convergent subsequence.

Theorem: Let S be a non-empty, bounded and closed-convex subset of the Banach space $\mathbb{X}$ and let $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: S \to \mathbb{X}$ are two operators satisfying:

1. $\mathbb{A}$ is Lipschitz with a lipschitz constant α,
2. $\mathbb{B}$ is completely continuous, and
3. $\mathbb{A}x\mathbb{B}x \in S$ for all $x \in S$, and
4. $\alpha M < 1$ where $M = \|\mathbb{B}(S)\|: \sup\{\|\mathbb{B}x\|: x \in S\}$.Then the operator equation $\mathbb{A}x\mathbb{B}x = x$ has a solution in S.

Existence of solutions

Definition: A mapping $\sigma: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Caratheodory if:

1. $t \to \sigma(t, x, y)$ is measurable for each $x, y \in \mathbb{R}$ and
2. $(x, y) \to \sigma(t, x, y)$is continuous almost everywhere for $t \in \mathbb{R}_+$.

Furthermore a Caratheodory function σ is $\mathcal{L}^1$ −Caratheodory if:

3. For each real number $r > 0$ there exists a function $\hbar_r \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$such that$|\sigma(t, x, y)| \le \hbar_r(t) \ \ a.e. \ t \in \mathbb{R}_+$ for all $x \in \mathbb{R}$ with $|x|_r \le r$ and $|y|_r \le$ r.

Finally a caratheodory function σ is $\mathcal{L}_X^1$ −caratheodory if:

4. There exists a function $\hbar \in \mathcal{L}^1(\mathbb{R}_+, \mathbb{R})$ such that $|\sigma(t, x, y)| \le \hbar(t), \ \ a.e. \ t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$

For convenience, the function $\hbar$ is referred to as a bound function for σ.

We will need the following hypothesis.

$(\boldsymbol{\mathcal{H}_1})$ The functions $\gamma, \mu: \mathbb{R}_+ \to \mathbb{R}$ are continuous.

$(\boldsymbol{\mathcal{H}_2})$ The function $f: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and bounded with bound $\mathbb{F} = sup_{(t,x(t),x(\gamma(t)))\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}} \left|f\left(t, x(t), x(\gamma(t))\right)\right|$ there exist a bounded function $k: \mathbb{R}_+ \to \mathbb{R}$ with bound K satisfying $\left|f\left(t, x(t), x(\gamma(t))\right) - f\left(t, y(t), y(\gamma(t))\right)\right|$ $\le k(t) max\{|x(t) - y(t)|, |x(\gamma(t)) - y(\gamma(t))|\}, \quad a.e. \ t \in \mathbb{R}_+$ for all $x, y \in \mathbb{R}$.

$(\boldsymbol{\mathcal{H}_3})$ The function $g: \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is satisfying caratheodory condition with continuous function $h(t): \mathbb{R}_+ \to \mathbb{R}$ such that $g(t, x, y) \le h(t) \ \forall t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$.

$(\boldsymbol{\mathcal{H}_4})$ The function $v: \mathbb{R}_+ \to \mathbb{R}$ defined by the formulas $v(t) = \int_0^t (t-s)h(s)\,ds$ is bounded on $\mathbb{R}_+$ and vanish at infinity, that is $\lim_{t\to\infty} v(t) = 0$.

Remark*:* Note that the $(\boldsymbol{\mathcal{H}_3}) - (\boldsymbol{\mathcal{H}_4})$ hold, then there exists a constant $K_1 > 0$ such that $K_1 = \sup\{v(t): t \in \mathbb{R}_+\}$

Lemma: Suppose that $\xi \in (0,1)$ and the function f, g satisfying SNDE (1.1) then x is the solution of the SNDE (1) if and only if it is the solution of integral equation

$$x(t) = \left[f\left(t, x(t), x(\gamma(t))\right)\right]\left[\int_0^t (t-s) g\left(s, x(s), x(\mu(s))\right) ds\right] \ t \in \mathbb{R}_+ \quad (3)$$

Proof: Integrating equation (1.1) of second order, we get

$$I\mathfrak{D}^2 \left[\frac{x(t)}{f\left(t, x(t), x(\gamma(t))\right)}\right]_0^t = I\left[g\left(s, x(s), x(\mu(s))\right)\right]$$

$$\Rightarrow \mathfrak{D}\left[\frac{x(t)}{f\left(t, x(t), x(\gamma(t))\right)}\right]_0^t = I\left[g\left(s, x(s), x(\mu(s))\right)\right]$$

$$\Rightarrow \mathfrak{D}\left[\frac{x(t)}{f\left(t, x(t), x(\gamma(t))\right)}\right]_0^t = I\left[g\left(s, x(s), x(\mu(s))\right)\right]$$

$$\Rightarrow \mathfrak{D}\left[\frac{x(t)}{f\left(t, x(t), x(\gamma(t))\right)}\right] = I\left[g\left(s, x(s), x(\mu(s))\right)\right],$$

Again integrating, we get

$$\left[\frac{x(t)}{f\left(t, x(t), x(\gamma(t))\right)}\right] = I^2\left[g\left(s, x(s), x(\mu(s))\right)\right]$$

$$x(t) = \left[f\left(t, x(t), x(\gamma(t))\right)\right]\left[\int_0^t \int_0^t g\left(s, x(s), x(\mu(s))\right) ds\right]$$

$$x(t) = \left[f\left(t, x(t), x(\gamma(t))\right)\right] \frac{1}{(2-1)!} \int_0^t (t-s) g\left(s, x(s), x(\mu(s))\right) ds$$

$$x(t) = \left[f\left(t, x(t), x(\gamma(t))\right)\right]\left[\int_0^t (t-s) g\left(s, x(s), x(\mu(s))\right) ds\right], \ \ t \in \mathbb{R}_+$$

Since $\int_0^t f(t)dt^n = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s)ds$, Where $n = 0,1,2,3, \dots \dots \dots$.

Conversely differentiate (3.1) w.r.to t, we get,

$$\mathfrak{D}\left[\frac{x(t)}{f\left(t, x(t), x\big(\gamma(t)\big)\right)}\right] = \mathfrak{D}I(t-s)g\left(s, x(t), x\big(\mu(t)\big)\right)$$

$$\mathfrak{D}\left[\frac{x(t)}{f\left(t, x(t), x\big(\gamma(t)\big)\right)}\right] = (t-s)g\left(s, x(t), x\big(\mu(t)\big)\right)$$

Again differentiating, we get,

$$\mathfrak{D}^2\left[\frac{x(t)}{f\left(t, x(t), x\big(\gamma(t)\big)\right)}\right] = 1g\left(s, x(t), x\big(\mu(t)\big)\right)$$

$$\mathfrak{D}^2\left[\frac{x(t)}{f\left(t, x(t), x\big(\gamma(t)\big)\right)}\right] = g\left(s, x(t), x\big(\mu(t)\big)\right)$$

Theorem: Assume that condition ($\boldsymbol{\mathcal{H}_1}$-$\boldsymbol{\mathcal{H}_4}$) hold. Further if $KK_1 < 1$, where K_1 is defined in remark (3.1). Then SNDE (1.1) has a solution in the space $\mathcal{AC}(\mathbb{R}_+, \mathbb{R})$, moreover solution of (1.1) are locally attractive on $\mathbb{R}_+$.

Proof: By a solution of SNDE (1.1) we mean a continuous function $x: \mathbb{R}_+ \to \mathbb{R}$ that satisfies SNDE (1) on $\mathbb{R}_+$. Set $\mathbb{X} = \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ and define a subset $\overline{B_r(0)}$ of $\mathbb{X}$ as $\overline{B_r(0)} = \{x \in \mathbb{X}: \|x\| \le r\}$.where r satisfies the inequality, $\mathbb{F}\mathcal{K}_1 \le r$.

Let $\mathbb{X} = \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ be Banach algebra of all absolutely continuous real-valued function on $\mathbb{R}_+$ with the norm $\|x\| = sup|x(t)|, t \in \mathbb{R}_+$ (2)

We shall obtain the solution of SNDE (1) under some suitable conditions involved in (1). Now the SNDE (1) is equivalent to the SNIE

$$x(t) = \left[f\left(t, x(t), x\big(\gamma(t)\big)\right)\right]\left[\int_0^t (t-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds\right]$$

Let us define the two mappings $\mathbb{A}: \mathbb{X} \to \mathbb{X}$ and $\mathbb{B}: \overline{B_r(0)} \to \mathbb{X}$ by

$$\mathbb{A}x(t) = f\left(t, x(t), x\big(\gamma(t)\big)\right), t \in \mathbb{R}_+ \tag{3}$$

$$\mathbb{B}x(t) = \int_0^t (t-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds\ , t \in \mathbb{R}_+ \tag{4}$$

Thus from the SNDE (1.1), we obtain the operator equation as follows:

$$x(t) = \mathbb{A}x(t)\mathbb{B}x(t)\ , t \in \mathbb{R}_+ \tag{5}$$

If the operator $\mathbb{A}$ and $\mathbb{B}$ satisfy all the hypothesis of theorem (3), then the operator equation (5) has a solution on $\overline{B_r(0)}$.

Step I: Firstly we show that $\mathbb{A}$ is Lipschitz on $\overline{B_r(0)}$ Let $x, y \in \overline{B_r(0)}$; then

$$|\mathbb{A}x(t) - \mathbb{A}y(t)| \le \left|f\left(t, x(t), x\big(\gamma(t)\big)\right) - f\left(t, y(t), y\big(\gamma(t)\big)\right)\right| \le k(t)max\{|x(t) - y(t)|, \left|x\big(\gamma(t)\big) - y\big(\gamma(t)\big)\right|\}$$

$\le k(t)|x(t) - y(t)|$ for all $t \in \mathbb{R}_+$

Taking suprimum over t we get,

$\|\mathbb{A}x - \mathbb{A}y\| \le \|K\|\|x - y\|$ for all $x, y \in \overline{B_r(0)}$

Thus, $\mathbb{A}$ is Lipchitz on $\overline{B_r(0)}$ with Lipschitz constant K.

Step II: Secondly we show that $\mathbb{B}$ is completely continuous operator on $\overline{B_r(0)}$ using standard argument such as those in Granas at [4], it can be shown that $\mathbb{B}$ is continuous operator on $\overline{B_r(0)}$. To do this, let us fix arbitrary $\epsilon > 0$ and take $x, y \in \overline{B_r(0)}$ such that $\|x - y\| \le \epsilon$.

$$|\mathbb{B}x(t) - \mathbb{B}y(t)| = \left|\int_0^t (t-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds - \int_0^t (t-s)g\left(s, y(s), y\big(\mu(s)\big)\right)ds\right|$$
$$\le \left|\int_0^t (t-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds\right| + \left|\int_0^t (t-s)g\left(s, y(s), y\big(\mu(s)\big)\right)ds\right|$$
$$\le \int_0^t (t-s)h(s)\,ds + \int_0^t (t-s)h(s)\,ds \le 2\int_0^t (t-s)h(s)\,ds$$

$\le 2v(t)$,As $v(t) \le \frac{\epsilon}{2}$, $|\mathbb{B}x(t) - \mathbb{B}y(t)| \le \epsilon$.

Thus $\mathbb{B}$ is continuous.

Now we will show that $\mathbb{B}\big(\overline{B_r(0)}\big)$ is uniformly bounded and equicontinuous set in $\overline{B_r(0)}$. Since $g\left(t, x(t), x\big(\mu(t)\big)\right)$ is $\mathcal{L}^1_{\mathbb{X}}$ − caratheodary, we have

$$|\mathbb{B}x(t)| = \left|\int_0^t (t-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds\right| \le \int_0^t (t-s)\left|g\left(s, x(s), x\big(\mu(s)\big)\right)\right|ds \le \int_0^t (t-s)h(s)ds \le v(t)$$

Taking suprimum over t, we obtain, $\|\mathbb{B}x\| \le K_1$ for all $x \in \overline{B_r(0)}$,

Where, $K_1 = sup_{t\in\mathbb{R}_+}\{v(t)\}$. This shows that $\mathbb{B}\big(\overline{B_r(0)}\big)$ is uniformly bounded set in $\mathbb{X}$.

To show that $\mathbb{B}\big(\overline{B_r(0)}\big)$ is an equicontinuous set, let $t_1, t_2 \in \mathbb{R}_+$ be arbitrary. Then for any $x \in \overline{B_r(0)}$,

$$|\mathbb{B}x(t_2) - \mathbb{B}x(t_1)| = \left|\int_0^{t_2} (t_2-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds - \int_0^{t_1} (t_1-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds\right|$$

$$\le \left|\int_0^{t_2} (t_2-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds - \int_0^{t_2} (t_1-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds\right|$$
$$+ \left|\int_0^{t_2} (t_1-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds - \int_0^{t_1} (t_1-s)g\left(s, x(s), x\big(\mu(s)\big)\right)ds\right|$$

$$\le \left|\int_0^{t_2} (t_2-s)h(s)ds - \int_0^{t_2} (t_1-s)h(s)ds\right| + \left|\int_0^{t_2} (t_1-s)h(s)ds - \int_0^{t_1} (t_1-s)h(s)ds\right|$$

$$\le \|h\|_{\mathcal{L}^1}\left\{\left|\int_0^{t_2}[(t_2-s) - (t_1-s)]ds\right| + \left|\int_{t_1}^{t_2}(t_1-s)ds\right|\right\}$$

$$\leq \|\mathrm{h}\|_{\mathcal{L}^1}\left\{\left|\left[-\frac{(t_2-s)^2}{2}\right]_0^{t_2}-\left[-\frac{(t_1-s)^2}{2}\right]_0^{t_2}\right| + \left|\left[-\frac{(t_1-s)^2}{2}\right]_{t_1}^{t_2}\right|\right\}$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{2}\left\{\begin{matrix}|-[(t_2-t_2)^2-(t_2-0)^2]-[-(t_1-t_2)^2+(t_1-0)^2]|+ \\ [-(t_1-t_2)^2+(t_1-t_1)^2]\end{matrix}\right\}$$

$$\leq \frac{\|h\|_{\mathcal{L}^1}}{2}\{|-(t_1)^2+(t_2)^2+(t_1-t_2)^2|+|-(t_1-t_2)^2|\}$$

The right hand side of the above inequality doesn't depend on X and tends to zero.

Therefore $|\mathbb{B}x(t_2)-\mathbb{B}x(t_1)| \to 0$ as $t_1 \to t_2$.
Hence, $\mathbb{B}\left(\overline{B_r(0)}\right)$ is an equicontinuous set and so $\mathbb{B}\left(\overline{B_r(0)}\right)$ is relatively compact by Arzela-Ascoli theorem. As a consequence, $\mathbb{B}$ is compact and continuous operator on $\left(\overline{B_r(0)}\right)$.

Thus $\mathbb{B}$ is completely continuous on $\left(\overline{B_r(0)}\right)$.

Step III: To show $x = \mathbb{A}x\mathbb{B}y \Rightarrow x \in \overline{B_r(0)}, \forall y \in \overline{B_r(0)}$

Let $x \in \mathbb{X}$, and $y \in \overline{B_r(0)}$ such that $x = \mathbb{A}x\mathbb{B}x$, By assumptions $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$

$$|x(t)| = |\mathbb{A}x(t)\mathbb{B}x(t)| \leq |\mathbb{A}x(t)||\mathbb{B}x(t)|$$

$$\leq \left|f\left(t, x(t), x(\gamma(t))\right)\right|\left|\int_0^t (t-s)g\left(s, x(s), x(\mu(s))\right)ds\right|$$

$$\leq \left|f\left(t, x(t), x(\gamma(t))\right)\right|\int_0^t (t-s)\left|g\left(s, x(s), x(\mu(s))\right)\right|ds$$

$$\leq \mathbb{F}\int_0^t (t-s)h(s)ds \leq \mathbb{F}v(t)$$

Taking supremum over t on $\mathbb{R}_+$, we obtain $\|\mathbb{A}x\mathbb{B}x\| \leq \mathbb{F}\mathcal{K}_1$, $\forall x \in \overline{B_r(0)}$

That is we have, $\|x\| = \|\mathbb{A}x\mathbb{B}x\| \leq r, \forall x \in \overline{B_r(0)}$.
Hence assumption (c) of theorem (2.3) is proved.

Step IV: Also we have

$$M = \left\|\mathbb{B}\left(\overline{B_r(0)}\right)\right\| = sup\left\{\|\mathbb{B}x\| : x \in \left(\overline{B_r(0)}\right)\right\}$$

$$= sup\left\{\begin{matrix} sup_{t\in\mathbb{R}_+}\left[\int_0^t (t-s)g\left(s, x(s), x(\mu(s))\right)ds\right] \\ : x \in \left(\overline{B_r(0)}\right)\end{matrix}\right\}$$

$$\leq sup\left\{sup_{t\in\mathbb{R}_+}\left[\int_0^t (t-s)h(s)ds\right] : x \in \left(\overline{B_r(0)}\right)\right\}$$

$$\leq sup\left\{sup_{t\in\mathbb{R}_+}[v(t)] : x \in \left(\overline{B_r(0)}\right)\right\} \leq K_1$$

and therefore $MK = KK_1 < 1$
Thus the condition (d) of theorem (2.3) is satisfied.

Hence all the conditions of theorem (2.3) are satisfied and therefore the operator equation $\mathbb{A}x\mathbb{B}x = x$ has a solution in $\left(\overline{B_r(0)}\right)$. As a result, the SNDE (1.1) has a solution defined on $\mathbb{R}_+$. **Step V:** Finally we show the locally attractivity of the solution for SNDE (1.1). Let x and y be two solutions of SNDE (1.1) in $\left(\overline{B_r(0)}\right)$ defined on $\mathbb{R}_+$. Then we have

$$|x(t)-y(t)| = \left|\begin{matrix}\left[f\left(t, x(t), x(\gamma(t))\right)\right]\left[\int_0^t (t-s)g\left(s, x(s), x(\mu(s))\right)ds\right] - \\ \left[f\left(t, y(t), y(\gamma(t))\right)\right]\left[\int_0^t (t-s)g\left(s, y(s), y(\mu(s))\right)ds\right]\end{matrix}\right|$$

$$\leq \left|\left[f\left(t, x(t), x(\gamma(t))\right)\right]\left[\int_0^t (t-s)g\left(s, x(s), x(\mu(s))\right)ds\right]\right| + \left|\left[f\left(t, y(t), y(\gamma(t))\right)\right]\left[\int_0^t (t-s)g\left(s, y(s), y(\mu(s))\right)ds\right]\right|$$

$$\leq \left|f\left(t, x(t), x(\gamma(t))\right)\right|\int_0^t (t-s)\left|g\left(s, x(s), x(\mu(s))\right)\right|ds + \left|f\left(t, y(t), y(\gamma(t))\right)\right|\int_0^t (t-s)\left|g\left(s, y(s), y(\mu(s))\right)\right|ds$$

$$\leq \mathbb{F}\left\{\int_0^t (t-s)h(s)\,ds\right\} + \mathbb{F}\left\{\int_0^t (t-s)h(s)\,ds\right\}$$

$$\leq 2\mathbb{F}\int_0^t (t-s)h(s)\,ds \leq 2\mathbb{F}[v(t)]$$

Since $\lim_{t\to\infty} v(t) = 0$ for $\epsilon > 0$, there is real number $\mathbb{T} > 0$ such that

$v(t) \leq \frac{\epsilon}{2\mathbb{F}}$ for all $t \geq \mathbb{T}$. Then from above inequality it follows that

$|x(t)-y(t)| < \epsilon$ for all $t \geq \mathbb{T}$. This completes the proof.

Existence of extremal solutions

A closed and non-empty set $\mathbb{K}$ in a Banach Algebra $\mathbb{X}$ is called a cone if

1. $\mathbb{K} + \mathbb{K} \subseteq \mathbb{K}$
2. $\lambda\mathbb{K} \subseteq \mathbb{K}$ for $\lambda \in \mathbb{R}, \lambda \geq 0$
3. $\{-\mathbb{K}\} \cap \mathbb{K} = 0$where 0 is the zero element of $\mathbb{X}$.
4. and is called positive cone if
5. $\mathbb{K} \circ \mathbb{K} \subseteq \mathbb{K}$

And the notation $\circ$ is a multiplication composition in $\mathbb{X}$
We introduce an order relation $\leq$ in $\mathbb{X}$ as follows.

Let $x, y \in \mathbb{X}$ then $x \leq y$ if and only if $y - x \in \mathbb{K}$. A cone $\mathbb{K}$ is called normal if the norm $\|\cdot\|$ is monotone increasing on $\mathbb{K}$. It is known that if the cone $\mathbb{K}$ is normal in $\mathbb{X}$ then every order-bounded set in $\mathbb{X}$ is norm-bounded set in $\mathbb{X}$.

We equip the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$ of continuous real valued function on $\mathbb{R}_+$ with the order relation $\leq$ with the help of cone defined by,

$$\mathbb{K} = \{x \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}) : x(t) \geq 0\ \forall t \in \mathbb{R}_+\} \tag{4.1}$$

We well known that the cone $\mathbb{K}$ is normal and positive in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$. As a result of positivity of the cone $\mathbb{K}$ we have:

Lemma[2]: Let $p_1, p_2, q_1, q_2 \in \mathbb{K}$ be such that $p_1 \leq q_1$ and $p_2 \leq q_2$then $p_1 p_2 \leq q_1 q_2$.

For any $p, q \in \mathbb{X} = \mathcal{C}(\mathbb{R}_+, \mathbb{R}), p \leq q$ the order interval $[p, q]$ is a set in $\mathbb{X}$ given by,

$$[p, q] = \{x \in \mathbb{X} : p \leq x \leq q\} \tag{4.2}$$

Definition[2]: A mapping $G : [p, q] \to \mathbb{X}$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Gx \leq Gy$ for all $x, y \in [p, q]$.

For proving the existence of extremal solutions of the equations (1.1) under certain monotonicity conditions by using following fixed pint theorem of Dhage [2]

Theorem 4.1 [2]: Let $\mathbb{K}$ be a cone in Banach Algebra $\mathbb{X}$ and let $[p, q] \in \mathbb{X}$. Suppose that $\mathbb{A}, \mathbb{B}: [p, q] \to \mathbb{K}$ are two operators such that

1. $\mathbb{A}$ is a Lipschitz with Lipschitz constant α,
2. $\mathbb{B}$ is completely continuous,
3. $\mathbb{A}x\mathbb{B}x \in [p, q]$ for each $x \in [p, q]$ and
4. $\mathbb{A}$and$\mathbb{B}$ are nondecreasing.

Further if the cone $\mathbb{K}$ is normal and positive then the operator equation $\mathbb{A}x\mathbb{B}x = x$ has the least and greatest positive solution in $[p, q]$ whenever $\alpha M < 1$, where $M = \|\mathbb{B}([p, q])\| = sup\{\|\mathbb{B}x\|: x \in [p, q]\}$

We need following definitions and hypothesis for existence the extremal solution of second order nonlinear differential equation (SNDE) (1.1)

Definition: A function $p \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ is called a **lower solution** of the SNDE (1.1) on $\mathbb{R}_+$ if the function $t \to \frac{p(t)}{f(t, p(t), p(\gamma(t)))}$ is continuous and

$$\left.\begin{array}{r}\mathfrak{D}^2\left[\dfrac{p(t)}{f\left(t, p(t), p(\gamma(t))\right)}\right] \le g\left(t, p(t), p(\mu(t))\right), a.e., t \in \mathbb{R}_+ \\ x(0) = 0\end{array}\right\}$$

Again a function $q \in \mathcal{AC}(\mathbb{R}_+, \mathbb{R})$ is called an upper solution of the SNDE (2.1.2) on $\mathbb{R}_+$ if the function $t \to \frac{q(t)}{f(t, q(t), q(\gamma(t)))}$ is continuous and

$$\left.\begin{array}{r}\mathfrak{D}^2\left[\dfrac{q(t)}{f\left(t, q(t), q(\gamma(t))\right)}\right] \ge g\left(t, q(t), q(\mu(t))\right), a.e., t \in \mathbb{R}_+ \\ x(0) = 0\end{array}\right\}$$

Definition: A solution x_M of the SNDE (1.1) is said to be maximal if for any other solution x to SNDE (1.1) one has $x(t) \le x_M(t)$ for all $t \in \mathbb{R}_+$. Again a solution x_M of the SNDE (1.1) is said to be minimal if $x_M(t) \le x(t)$ for all $t \in \mathbb{R}_+$ where x is any solution of the SNDE (1.1) on $\mathbb{R}_+$.

Definition: (Caratheodory Case)

We consider the following set of hypothesis:

$\mathfrak{B}5$) $g$is Caratheodory.

$\mathfrak{B}6$) The functions $f\left(t, x(t), x(\gamma(t))\right)$ and $g\left(t, x(t), x(\mu(t))\right)$ are non-decreasing in x almost everywhere for $t \in \mathbb{R}_+$.

$\mathfrak{B}7$) The SNDE (1.1) has a lower solution p and an upper solution q on $\mathbb{R}_+$ with $p \le q$.

$\mathfrak{B}8$) The function $l: \mathbb{R}_+ \to \mathbb{R}$ defined by,

$l(t) = \left|g\left(t, p(t), p(\mu(t))\right)\right| + \left|g\left(t, q(t), q(\mu(t))\right)\right|$ is Lebesgue measurable.

Remark: Assume that $(\mathfrak{B}6 - \mathfrak{B}8)$ hold. Then

$\left|g\left(t, x(t), x(\mu(t))\right)\right| \le l(t), a.e. t \in \mathbb{R}_+$, for all $x \in [p, q]$.

Theorem: Suppose that the assumptions **($\mathcal{H}_5$)-($\mathcal{H}_8$)** and $(\mathfrak{B}5 - \mathfrak{B}8)$ holds and l is given in remark (4.1) further if $kT\|l\|_{\mathcal{L}^1} \le 1$ then SNDE (1.1) has a minimal and maximal positive solution on $\mathbb{R}_+$.

Proof: Now SNDE (1.1) is equivalent to IE (3.1) $\mathbb{R}_+$. Let $\mathbb{X} = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ and define an order relation "≤" by the cone $\mathbb{K}$ given by (4.1). Clearly $\mathbb{K}$ is a normal cone in $\mathbb{X}$. Define two operators $\mathbb{A}$ and $\mathbb{B}$ on $\mathbb{X}$ by (3.3) and (3.4) respectively. Then IE (3.1) is transformed into an operator equation $\mathbb{A}x\mathbb{B}x = x$ in a Banach algebra $\mathbb{X}$. Notice that $(\mathfrak{B}5)$ implies $\mathbb{A}, \mathbb{B}: [p, q] \to \mathbb{K}$ Since the cone $\mathbb{K}$ in $\mathbb{X}$ is normal, $[p, q]$is a norm bounded set in $\mathbb{X}$. Now it is shown, as in the proof of Theorem (3.1), that $\mathbb{A}$ is a Lipschitz with a Lipschitz constant K and $\mathbb{B}$ is completely continuous operator on $[p, q]$. Again the hypothesis $(\mathfrak{B}6)$ implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing on $[p, q]$. To see this, let $x, y \in [p, q]$ be such that $x \le y$. Then by $(\mathfrak{B}6)$

$$\mathbb{A}x(t) = f\left(t, x(t), x(\gamma(t))\right) \le f\left(t, y(t), y(\gamma(t))\right) = \mathbb{A}y(t), \forall t \in \mathbb{R}_+$$

Similarly,

$$\mathbb{B}x(t) = \int_0^t (t-s) g\left(s, x(s), x(\mu(s))\right) ds \le \int_0^t (t-s) g\left(s, y(s), y(\mu(s))\right) ds \le \mathbb{B}y(t), \forall t \in \mathbb{R}_+$$

Implies that $\mathbb{A}$ and $\mathbb{B}$ are non-decreasing operators on $[p, q]$. Again definition (2.5.2.1) and hypothesis $(\mathfrak{B}7)$ implies that

$$p(t) \le f\left(t, p(t), p(\gamma(t))\right) \int_0^t (t-s) g\left(s, p(s), p(\mu(s))\right) ds$$

$$\le f\left(t, x(t), x(\gamma(t))\right) \int_0^t (t-s) g\left(s, x(s), x(\mu(s))\right) ds$$

$$\le f\left(t, q(t), q(\gamma(t))\right) \int_0^t (t-s) g\left(s, q(s), q(\mu(s))\right) ds$$

$$\le q(t), \forall t \in \mathbb{R}_+ \text{and} x \in [p, q]$$

As a result $p(t) \le \mathbb{A}x(t)\mathbb{B}x(t) \le q(t), \forall t \in \mathbb{R}_+$ and $x \in [p, q]$

Hence $\mathbb{A}x\mathbb{B}x \in [p, q], \forall x \in [p, q]$

Again $M = \|\mathbb{B}([p, q])\| = sup\{\|\mathbb{B}x\|: x \in [p, q]\}$

$$\le sup\left\{sup_{t\in\mathbb{R}_+} \int_0^t \left|(t-s) g\left(s, x(s), x(\mu(s))\right) ds\right| : x \in [p, q]\right\}$$

$$\le sup\left\{T sup_{t\in\mathbb{R}_+} \int_0^t \left|g\left(s, x(s), x(\mu(s))\right) ds\right| : x \in [p, q]\right\}$$

$\le T \int_0^t l(s)\, ds \le T\|l\|_{\mathcal{L}^1}$,Since $\alpha M \le KT\|l\|_{\mathcal{L}^1} \le 1$

We apply theorem (4.1) to the operator equation $\mathbb{A}x\mathbb{B}x = x$ to yield that the SNDE (1.1) has minimum and maximum positive solution on $\mathbb{R}_+$.

This completes the proof.

Example

Example: Consider the following second order nonlinear differential equation of type (1.1)

$$\left.\begin{array}{r}\mathfrak{D}^2\left\{\dfrac{x(t)}{cost\left[\frac{x(t)}{1-x(t)} + e^{-t}\right]}\right\} = \dfrac{t}{t^5[1 + x(4t)]} \\ x(0) = 0, \forall t \in \mathbb{R}_+\end{array}\right\} \quad (1)$$

Solution: Here,

$f\left(t, x(t), x(\gamma(t))\right) = cost\left[\frac{x(t)}{1-x(t)} + e^{-t}\right], g\left(t, x(t), x(\mu(t))\right) = \frac{t}{t^5[1+x(4t)]}$

and $\gamma = t, \mu = 4t$

$(\mathcal{H}_1)$ The functions $\gamma = t, \mu = 4t$ are continuous on $\mathbb{R}_+$.

$(\mathcal{H}_2)$ Let, $\left|f\left(t,x(t),x(\gamma(t))\right)-f\left(t,y(t),y(\gamma(t))\right)\right|$

$$=\left|\left\{cost\left[\frac{x(t)}{1-x(t)}+e^{-t}\right]\right\}-\left\{cost\left[\frac{y(t)}{1-y(t)}+e^{-t}\right]\right\}\right|$$

$$=\left|cost\left[\frac{x(t)}{1-x(t)}-\frac{y(t)}{1-y(t)}\right]\right|$$

$$\leq |cost|\left|\frac{x(t)y(t)+x(t)-y(t)-x(t)y(t)}{x(t)y(t)-x(t)-y(t)+1}\right|$$

$$\leq |cost||x(t)-y(t)|$$

$\leq k(t)|x(t)-y(t)|$

$\leq K|x(t)-y(t)|$, Since $K=cost$ say which has bound K bounded on $\mathbb{R}_+$.

$(\mathcal{H}_3)$ Take $h(t)=\frac{1}{t^4}$, it is continuous on $\mathbb{R}_+$.

$\therefore g\left(t,x(t),x(\mu(t))\right)\leq h(t)$, That is $\frac{t}{t^5[1+x(4t)]}\leq\frac{1}{t^4}$

$(\mathcal{H}_4)$Now $v(t)=\int_0^t(t-s)h(s)ds=\int_0^t(t-s)s^{-4}\,ds=\left[(t-s)\frac{s^{-3}}{-3}\right]_0^t-\int_0^t(-1)\left(\frac{s^{-3}}{-3}\right)ds$

$$=\frac{1}{-3}\{(t-t)t^{-3}-(t-0)0^{-3}\}-\frac{1}{3}\left\{\left[\frac{s^{-2}}{-2}\right]_0^t\right\}=\frac{1}{-3}(0)-\left\{\frac{1}{-6}(t^{-2}-0^{-2})\right\}$$

$$=\frac{1}{6t^2}\to 0\ as\ t\to\infty$$

It follows that all the conditions $(\mathcal{H}_1)-(\mathcal{H}_4)$ satisfied.
Thus by theorem (3.1) above problem has a locally attractive solution on R_+.

CONCLUSION

In this paper we have studied the existence the lacally attractive solution and its extremal solution of second order nonlinear differential equation. The result has been obtained by using hybrid fixed point theorem for two operators in Banach space due to Dhage. The main result is well illustrated with the help of example.

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How to cite this article:

Karande B. D and Yachawad S. S (2018) 'Existence the Locally Attractive Solution For A Second Order Nonlinear Differential Equation in Banach Space', *International Journal of Current Advanced Research,* 07(5), pp. 12309-12314.
DOI: http://dx.doi.org/10.24327/ijcar.2018.12314.2159

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