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Research Article

OSEEN'S CORRECTION TO STOKES DRAG REVISITED

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ABSTRACT

In the present review article, intensive survey on Oseen's approximation, Oseen's correction and solution to Oseen's equation is presented. Further, general expressions of Oseen's correction to Stokes drag on axially symmetric particle placed in uniform stream parallel to axis of symmetry (longitudinal flow) and perpendicular to axis of symmetry (transverse flow) are derived with the help of DS-conjecture(Datta and Srivastava, 1999). These results are applied to sphere and spheroids to evaluate the Oseen's drag corrected up to first order terms in Reynolds number Re. Some practical applications are also highlighted.

INTRODUCTION

To understand the dispersion of fine particles suspended in many geophysical and physiological flows, a detailed knowledge of the forces acting on them is required. The problem of great importance in the hydrodynamics of low Reynolds number flows is the drag or resistance experienced by a particle moving uniformly through an infinite fluid. Since the appearance of Stokes's approximate solution for the flow of a viscous fluid past a sphere (Stokes, 1851), very well known as Stokes law, numerous attempts have been made, both to generalize the problem by changing the shape of the body, and to improve the calculation by including the effect of the inertia terms which were neglected in the original calculation. Oseen (1927) tackled this type of problem involving the correction to Stokes drag extensively. Oseen provided solutions for the flow past various bodies at small Reynolds number 'R' and calculated the force to the first order in R, one term more than would be given by the Stokes approximation. By the inclusion of the effect of the inertia terms, Oseen improved the flow picture far from the body where the Stokes approximation is inadequate, but near the body the difference between the two solutions is of an order of smallness which is outside the accuracy of either approximation.

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Oseen's calculation for the force thus requires some further justification, for flow past a sphere, by the work of Kaplun (1957), Kaplun and Lagerstrom(1957) and Proudman and Pearson(1957). Oseen failed to calculate correctly the velocity field, his result for the drag on the sphere, namely

$$D = D_0[\ 1 + (3/8)R], \qquad (1)$$

where D_0 is the Stokes drag, is in fact valid because the correction to the velocity field makes no contribution to the total force on the sphere. Almost similar problem has been considered by Chang (1960) for the axially symmetric Stokes flow of a conducting fluid past a body of revolution in the presence of a uniform magnetic field. An equation identical to that cited above, except that the dimensionless Hartmann number, M, appears in place of the Reynolds number, R. Chang(1960) gave a formal proof of this relation based on 'matching' the fundamental solution of the Stokes equations to the fundamental solution of his magneto-hydrodynamics differential equation. These fundamental solutions correspond physically to the fields resulting from a point force concentrated at the origin, and are termed 'inner' and 'outer' solutions, respectively. The ideas used in these papers were based on the work of Lagerstrom and Cole(1955) and Proudman and Pearson(1957). The structure of Oseen's equation and its fundamental solution are extremely similar to those for the corresponding magneto hydrodynamics. They do, however, differ in one important respect. The first-order Stokes(inner) equation(Proudman and Pearson, 1957, eq. 3.40), which eventually gives rise to the drag term of O(R), contains an inhomogeneous forcing term. This is lacking in the analogous first-order inner equation of Chang (1960), equation (6b), which is the source of the drag term of order O(M) in the magneto hydrodynamic problem. Moreover, Proudman and Pearson (1957) have fully discussed the 'matching' of the Oseen and Stokes equations for the particular case of a spherical particle. Perhaps for these reasons, Brenner (1961) skipped the proof of his own formula (given in eq. 8) regarding Oseen's drag on axially symmetric bodies. According to him, the combined work of Proudman and Pearson (1957) and Chang (1960) assured the existence of a general relation of the form of equation 8. It is interesting to note here that Chang's (1960) result is restricted to axially symmetric flows because of the requirement that there be sufficient symmetry to preclude the existence of an electric field. While, on the other hand, Brenner's (1961) result is limited only by the requirement that the Stokes drag on the particle (and thus the Oseen drag) be parallel to its direction of motion.

Filon (1928) provided the second approximation to the "Oseen" solution for the motion of viscous fluid. Southwell and Squire (1934) modified the Oseen's approximate equation for the motion in two dimensions of a viscous incompressible fluid possessing finite viscosity. Garstang (1933) discussed the solution for flow past a spinning sphere using Oseen's equations. Meksyn (1937) obtained the solution of Oseen's equations for an inclined elliptic cylinder in a viscous fluid. Westervelt (1953) proved that a linearization of the hydrodynamic equations in the manner of Oseen does not lead to a description of the steady flow phenomena which occur in connection with oscillatory flow past solid obstacles. Imai (1954) developed a general method of solving Oseen's linearized equations for a two-dimensional steady flow of a viscous fluid past an arbitrary cylindrical body. Wilkinson (1955) proved that the Oseen equations have a simple solution when the body is in the form of a paraboloid of elliptic section in a stream of viscous, incompressible fluid which at infinity has uniform velocity in the direction of the axis. Tyusei(1955) made detailed theoretical discussion on the steady flow of an incompressible viscous fluid past a prolate or oblate spheroid by use of an exact solution of Oseen's equations of motion including a circular disc as a special case. Kawaguti (1956) studied Oseen's approximation to the shear flow of a viscous fluid around a circular cylinderand compared with the Stokes approximation. Proudman and Pearson (1957) obtained higher order approximation to the flow past a sphere and a circular cylinder than those represented by the well known solutions of Stokes and Oseen by using perturbation technique. Hocking(1959) found the streaming motion of a viscous liquid past a circular disk, fixed normal to the stream on the basis of Oseen's approximation by taking distributions of singular solutions of the equations of motion over the surface of the disk. Kawaguti (1959) developed the simple relations which exist between the two-dimensional flow of a viscous fluid past a cylindrical obstacle and the flow past the same obstacle rotated by an angle π about its axis, within the accuracy of Oseen approximation. Bhatnagar (1961) studied the steady flow of a viscous fluid past a sphere at small Reynolds numbers using Oseen's approximation. Brenner (1961) found the Oseen's drag of a particle of arbitrary shape moving parallel to a principal axis of resistance through an unbounded fluid. Chester (1962) made an investigation into the validity of the Oseen equations, for incompressible, viscous flow past a body, as an approximation to the Navier-Stokes equations. He has shown that the Oseen's drag can be deduced simply from knowledge of the force on the body according to Stokes approximation. Chester(1962) generalized his analysis to include the magneto hydrodynamic effects when the fluid is conducting and the flow take place in the presence of a magnetic field.Viviand and Berger (1964)investigated the influence of inertia terms in the equations of motion on the properties of the base flow and near wake flow at very low Reynolds numbers by using Oseen's approximation and compared with previous results obtained by same authors for Stokes flow. Krasovitskaya *et al.*(1970) proposed Oseen correction on the basis of the Stokes law for calculating the settling of solid particles of powdered materials with enhanced accuracy in carrying out sedimentation analysis. Van Dyke (1970) extended the Goldstein's expansion of the Oseen drag of a sphere in powers of Reynolds number to 24 terms by using computer. He found that the convergence is limited by a simple pole at $R = -4.18172$. Yoshizawa(1970) studied the viscous flow past a semi-infinite flat plate by means of the second approximation of the Oseen-type successive approximation. Gautesen (1971a) studied the problem of two-dimensional, steady Oseen flow past a semi-infinite flat plate with two lines of concentrated force singularities located symmetrically with respect to the plane of plate. Gautesen (1971b) considered the problem of steady Oseen flow past a semi-infinite flat plate in the presence of a force singularity acting in an arbitrary direction at an arbitrary position. Yoshizawa(1972) studied laminar viscous flow past a semi-infinite flat plate placed parallel to a uniform stream by means of the second approximation of the modified Oseen-type successive approximation. Ranger(1972) written a note on Oseen flow. In which he has shown that Oseen flow can be constructed and derived quite simply as a fractional integral of the corresponding two-dimensional flow. Bruyatskii(1966) found the solution of Oseen's equations by applying the method of I.N. Vekua(1967). Chadwick and Zvirin(1974) provided the correction to the Stokes drag formula for a sphere in uniform horizontal flow of a continuous vertically stratified, incompressible fluid. Kaplan(1975) derived particular solutions of the equations for the vorticity for both two-dimensional and two-dimensional and axi-symmetrical Oseen flow. He then utilized the same for the solutions of the differential equations for the stream functions for the flow past a parabolic cylinder in two-dimensional flow and for the flow past a paraboloid of revolution in axi-symmetrical flow. Chwang and Wu(1976) analyzed the problem of a uniform transverse flow past a prolate spheroid of arbitrary aspect ratio at Reynolds numbers by the method of matched asymptotic expansions. They obtained drag formula for small values of $R_b=Ub/\nu$ and arbitrary values of $R_a= Ua/\nu$. For low Reynolds number R_a, this drag formula reduces to the Oberbeck(1876) result for Stokes flow past a spheroid, and it gives the Oseen drag for an infinitely long cylinder when R_a tends to infinity. Bentwich and Miloh (1978) presented the matched asymptotic expansion type of solution for the unsteady viscous incompressible flow past a sphere and applied unsteady Oseen equations. Yano and Kieda(1980) presented an approximate method for solving Oseen's linearized equations for a two-dimensional steady flow of incompressible viscous fluid past arbitrary cylindrical bodies at low Reynolds numbers. Sakurai and Fukuyu (1981) derived some general properties of the approximation under a few assumption to show how the results of usual Oseen approximation to incompressible flow be modified. Homentcovschi (1981) studied the Oseen's stationary motion of an incompressible viscous fluid past a slender body of revolution. Dziurda(1982) discussed the existence and uniqueness of the generalized solution of Oseen's equations. Olmstead and Majumdar(1983) considered the problem of steady, incompressible flow of a micropolar fluid in two dimensions. They introduced the convective operator for Oseen linearization and obtained the fundamental solution of the fundamental problem in explicit form under a certain restriction on the physical parameters of the problem. Weisenborn and Mazur (1984) presented a scheme to evaluate the Oseen drag on an infinite circular, at rest in a perpendicular uniform stationary flow. Mazur and Weisenborn (1984) calculated the Oseen drag experienced by a sphere by using the method of induced forces. Lee (1984) investigated the numerical and analytical study of drag on a sphere in Oseen's approximation. Wu and Mei (1986) studied the wake effect on drag factor in the axi-symmetric Oseen flow of the finite clusters of equally spaced spheres with same size.Lee and Leal(1986) evaluated the general solutions for both the Stokes and Oseen equationsin two dimensions expressed in terms of a boundary distribution of fundamental single and double layer singularities.Olmstead *et al*.(1986) investigated the drag experienced in a micropolar fluid by considering uniform streaming past a flat plate. Sakoh *et al*.(1987) dealt with the inverse problem for obtaining optimal body profiles having minimal viscous drag in two dimensional Oseen flow.Lopez and Rubo(1988) investigated the effect of including corrections to the Oseen hydrodynamic interaction in the rheological properties of polymer solutions. Dennis and Kocabiyik(1990) developed a general method of solving Oseen's linearized equations for two-dimensional steady flow of a viscous incompressible fluid past a cylinder in an unbounded field. Ranger(1990) introduced Burgers type linearization at small Reynolds number which converted the exact Navier-Stokes equations for the steady two-dimensional motion of an incompressible viscous fluid starting from a complex variable formulation in to a conformally invariant Oseen equation.Sasic and Sasic(1991) proposed a new method for solving the Oseen's equations and tested it on oblique flow around cylinder.Power and Power(1992) solved Oseen's system of equations by multiple reciprocity method. Palaniappan *et al*.(1992) wrote the expressions of velocity and pressure in Stokes flow and Oseen's flow in terms of biharmonic and harmonic functions. Weisenborn and Bosch(1993) evaluated the Oseen drag coefficient for a sphere at infinite Reynolds number analytically.Kuhimann and Ramachandra(1993) developed a biorthogonal series method to solve Oseen type flow problems. Lovalenti and Brady(1993) found that the expression for the hydrodynamic force is not simply an additive combination of the results from unsteady Stokes flow and Oseen flow and that the temporal decay to steady state for small but finite Re is always faster than the $t^{-1/2}$behaviour of unsteady Stokes flow. Weisenborn and Bosch(1995a) shown that the Oseen drag on an arbitrarily shaped object at infinite Reynolds number is the sum of two contributions.Weisenborn and Bosch(1995b) evaluated the Oseen drag on a sphere by employing the method of induced forces. Richardson and Power(1996) developed a 3D numerical model for the problem of creeping viscous flow past porous bodies based upon an integral equation formulation. They incorporated the inertial effects into the formulation of by use of Oseen approximation. Chadwick(1998) considered the uniform, steady, incompressible fluid flow in an unbounded domain past a fixed, closed body and provided Oseen velocity and pressure expansions in the far field for two and three dimensional flow by using two approaches.Ferreira *et al*.(1998) carried out a theoretical study of the transient sphere motion(under the influence of gravity) through an incompressible Newtonian fluid subject to an Oseen type drag relationship. Weisenborn and Bosch(1998) discussed a variational principal for the drag on moving sphere in Oseen flow. Chan and Chwang(2000) investigated the unsteady low Reynolds number flow of an incompressible viscous fluid past a singular forcelet analytically and derived new fundamental three-dimensional solutions for a concentrated impulsive force for the Stokes and Oseen equations. Titcombe *et al*.(2000) applied a hybrid asymptotic-numerical method to compute the drag coefficient on sphere and cylindrical body in Oseen region. Ikeda *et al*.(2001) considered Oseen's spiral flows for viscous incompressible fluid. Chadwick(2002) considered steady flow generated by a uniform velocity field past a fixed closed slender body whose major axis is aligned closely to the uniform stream direction under slip boundary condition. Nishida (2003) considered the Oseen's linearization equations in three dimensional exterior domains and propose a new numerical method based on fundamental solution based on fundamental solution method by numerical experiments. Amrouche and Razafison (2004) studied the nonhomogeneous Oseen equations in R^n and proved the existence and uniqueness result in weighted Sobolev spaces. Boulmezaoud and Razafison (2005) dealt with Oseen's equations and gave a class of existence, uniqueness and regularity results for both scalar and the vectorial equations. Burman *et al*.(2006) presented an extension of the continuous interior penalty method of Douglas and Dupont(1976) to Oseen's equations.Guenther and Thomann(2006) determined the fundamental solutions for the linearizations of Stokes and Oseen of the Navier-Stokes time dependent equations in two spatial dimensions.Chadwick(2006) gave the horseshoe vortex in Oseen flow as a constant spanwise distribution of lift oseenlets.Venkatalaxmi *et al*.(2007) proposed a new form of general solution of Oseen equations and examples of oseenlets are discussed. Padmavathi (2007) suggested a new general solution of Oseen equations and provided a necessary and sufficient condition for a divergence free vector to represent the velocity in an Oseen flow. Posta and Roubicek (2007) tackled the problem of optimal control of Navier-Stokes equations by Oseen approximation.Wang and Wen (2008) investigated the interactions between two identical spherical particles in Oseen flow for small Reynolds numbers using the method of fundamental solutions(MFS). Chadwick (2009) considered the problem of steady, uniform flow at a small angle of incidence to a slender body inOseen's approximation under slip boundary conditions. Zabarankin(2010) identified a class of generalized analytic functions from the Oseen equations in the axially symmetric case and obtained the generalized Cauchy integral formula used later on for the construction of a series representation for the region exterior to a sphere in axi-symmetric Oseen flow.Chadwick and Elmazuzi(2011) presented the oscillatory oseenlet solution for the velocity and pressure, and calculated the forces generated by them which are further shown to be oscillatory. Srivastava *et al*.(2013) studied the problem of Steady Oseen flow past deformed sphere by applying an analytic approach. Krasovitskaya *et al*.(1970) proposed a formula based on Oseen's correction for calculating the settling of solid particles of powdered materials with enhanced accuracy in carrying out sedimentation analysis. Dyer and Ohkawa(1992) have used the Oseen drag in acoustic levitation. These two works are the main practical applications of Oseen's correction which was not possible with the Stokes drag.

METHOD

Oseen's correction to Stokes drag.Let us consider the axially symmetric body of characteristic length L placed along its axis (x-axis, say) in a uniform stream U of viscous fluid of density ρ_1 and kinematic viscosity ν. When Reynolds number UL/ν is small, the steady motion of incompressible fluid around this axially symmetric body is governed by Stokes equations [Happel and Brenner, 1964],

$$\mathbf{0} = -\left(\frac{1}{\rho_1}\right)\text{grad } p + \nu \nabla^2 \mathbf{u}, \ \text{div}\mathbf{u} = 0, \tag{2}$$

subject to the no-slip boundary condition.

we have taken up the class of those axially symmetric bodies which possesses continuously turning tangent, placed in a uniform stream U along the axis of symmetry (which is x-axis), as well as constant radius 'b' of maximum circular cross-section at the mid of the body. This axi-symmetric body is obtained by the revolution of meridional plane curve (depicted in **figure 1a**) about axis of symmetry which obeys the following limitations:

1. Tangents at the points A, on the x-axis, must be vertical,
2. Tangents at the points B, on the y-axis, must be horizontal,
3. iii.The semi-transverse axis length 'b' must be fixed.

The point P on the curve is may be represented by the Cartesian coordinates (x,y) or polar coordinates (r,θ) respectively , PN and PM are the length of tangent and normal at the point P. The symbol R stands for the intercepting length of normal between the point on the curve and point on axis of symmetry and symbol α is the slope of normal PM which can be vary from 0 to π.

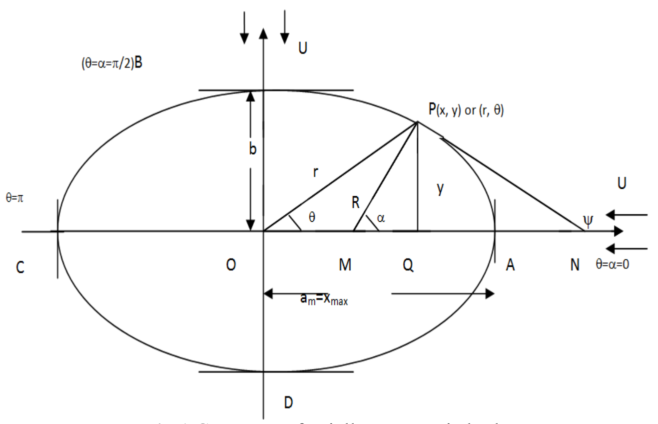

Fig 1 Geometry of axially symmetric body

Axial flow.The expression of Stokes drag on such type of axially symmetric bodies placed in axial flow (uniform flow parallel to the axis of symmetry) is given by [Datta and Srivastava, 1999]

$$F_x = \frac{1}{2}\frac{\lambda b^2}{h_x}, \text{ where } \lambda = 6\pi\mu U \tag{3}$$

$$\text{and } h_x = \left(\frac{3}{8}\right)\int_0^{\pi} R \sin^3 \alpha \, d\alpha . \tag{4}$$

where the suffix 'x' has been introduced to assert that the force is in the axial direction.

Transverse flow.The expression of Stokes drag on axially symmetric bodies placed in transverse flow(uniform flow perpendicular to the axis of symmetry) is given by[Datta and Srivastava, 1999]

$$F_y = \left(\frac{1}{2}\right)\frac{\lambda b^2}{h_y}, \text{ where } \lambda = 6\pi\mu U, \tag{5}$$

$$\text{and } h_y = \left(\frac{3}{16}\right)\int_0^{\pi} R\left(2\sin\alpha - \sin^3\alpha\right)d\alpha . \tag{6}$$

where the suffix 'y' has been introduced to assert that the force is in the transverse direction.

Now, on dividing (3) and (5), we get

$$\frac{F_x}{F_y} = \frac{h_y}{h_x} = \frac{1}{2}\frac{\int_0^{\pi} R\left(2\sin\alpha - \sin^3\alpha\right)d\alpha}{\int_0^{\pi} R\sin^3\alpha \, d\alpha} = K(say), \tag{7}$$

On applying Brenner's formula (Brenner, 1961), Oseen's correction to Stokes drag on a body placed in axial uniform flow, in general, may be written as

$$\frac{F}{F_x} = 1 + \frac{F_x}{16\pi\mu L U} R + O(R^2), \tag{9}$$

by using linear relationship between axial and transverse Stokes drag (8), equation (9) provide Oseen's correction to Stokes drag on a body placed in transverse uniform flow

$$\frac{F}{F_y} = K\frac{F}{F_x} = K\left[1 + \frac{F_x}{16\pi\mu L U} R + O(R^2)\right]. \tag{10}$$

where K is a real factor defined in (7) and R = ρ_1UL/μ, the particle Reynolds number.

Formulation of the problem

Let us consider the axially symmetric arbitrary body of characteristic length L placed along its axis (x-axis, say) in a uniform stream U of viscous fluid of density ρ_1 and kinematic viscosity ν, perpendicular to axis of symmetry. When particle Reynolds number UL/ν is small, the steady motion of incompressible fluid around this axially symmetric body is governed by Stokes equations [Happel and Brenner, 1964],

$$\mathbf{0} = -\left(\frac{1}{\rho_1}\right)\text{grad } p + \nu \nabla^2 \mathbf{u}, \ \text{div}\mathbf{u} = 0, \tag{11}$$

subject to the no-slip boundary condition. This equation is the reduced form of complete Navier-Stokes equations neglecting inertia term (u.grad)u which is unimportant in the vicinity of body where viscous term dominates(Stokes approximation). Solution of this equation (11), called Stokes law, 6πμUa, for a slowly moving sphere having radius 'a', is valid only in the vicinity of the body which breaks down at distance far away from the body. This breaks down of Stokes solution at far distance from the body is known as Whitehead's paradox. It was Oseen in 1910, who pointed out the origin of Whitehead's paradox and suggest a scheme for its resolution (see Oseen, 1927). In order to rectify the difficulty, Oseen went on to make the following additional observations.

In the limit where the particle Reynolds number $\rho_1 Ua/\mu \to 0$, Stokes approximation becomes invalid only when $r/a \to \infty$. But at such enormous distances, the local velocity **v** differs only imperceptibly from a uniform stream of velocity **U**. Thus, Oseen was inspired to suggest that the inertial term (u.grad) ucould be uniformly approximated by the term (U.grad)u. By such arguments, Oseen proposed that uniformly valid solutions of the problem of steady streaming flow past a body at small particle Reynolds numbers could be obtained by solving the linear equations

$$(\mathbf{U}.\text{grad})\mathbf{u} = -\left(\frac{1}{\rho_1}\right)\text{grad}\, p + \nu\nabla^2\mathbf{u},\ \text{div}\,\mathbf{u} = 0, \tag{12}$$

known as Oseen's equation. Oseen obtained an approximated solution of his equations for flow past a sphere, from which he obtained the Stokes drag formula[Happel and Brenner, page 44, eq.(2-6.5), 1964]

$$F = 6\pi\mu a U\left[1 + \frac{3}{8}N_{Re} + O\left(N_{Re}{}^2\right)\right], \tag{13}$$

where$N_{Re} = \rho_1 Ua/\mu$ is particle Reynolds number.We find the solution of these equations (12) for various axially symmetric bodies like sphere, prolate and oblate spheroid under no-slip boundary conditions by use of D-S conjecture (3), (5), followed by linear relationship (8) and Brenner's formula (9) valid for axial flow and its extension (10) for transverse flow.

Flow past sphere

Stokes drag on sphere having radius 'a' placed in uniform axial flow, with velocity U, parallel to axis of symmetry(x-axis) and very well known as Stokes law of resistance is given by(by utilizing DS conjecture 5 and 6, Datta and Srivastava, 1999)

$$F_x = F_y = 6\pi\mu U a. \text{(14a,b)}$$

From relation (14a, b) and (8), the value of K = 1. (15)

Now, the Oseen's correction as well as the solution of Oseen's equation (12) may be obtained for same sphere by substituting the value of K=1 and Stokes drag (14a,b) in Brenner's formula (9) and (10) as

$$\frac{F}{F_y} = \frac{F}{F_x} = 1 + \frac{3}{8}R + O\left(R^2\right), \tag{16}$$

where $R = \left(\frac{\rho U a}{\mu}\right)$ is particle Reynolds number. This Oseen's correction matches with that given by Oseen (1927) and Chester(1962).

Flow past a prolate spheroid

Stokes drag on prolate spheroid having semi-major axis length 'a' and semi-minor axis length 'b' placed in uniform velocity U, parallel to axis of symmetry(axial flow) and perpendicular to its axis of symmetry(transverse flow) is given as[by utilizing formulae 5 and 6, Datta and Srivastava, 1999]

$$F_x = 16\pi\mu U a e^3\left[-2e + \left(1+e^2\right)\ln\frac{1+e}{1-e}\right]^{-1}, \tag{17a}$$

$$F_y = 32\pi\mu U a e^3\left[2e + \left(3e^2 - 1\right)\ln\frac{1+e}{1-e}\right]^{-1}. \tag{17b}$$

By using (17a,b), from (8), the value of real factor 'K' comes out to be

$$K = \frac{1}{2}\left[2e + \left(3e^2-1\right)\ln\frac{1+e}{1-e}\right]\left[-2e + \left(1+e^2\right)\ln\frac{1+e}{1-e}\right]^{-1}$$

$$= 1 - \frac{1}{10}e^2 - \frac{8}{175}e^4 ..... \tag{18}$$

Now, from Brenner's formula (9) and (10), the Oseen's correction, with the use of real factor K(eq. 18) may be written as

$$\frac{F}{F_y} = K\frac{F}{F_x}$$

$$= \left[1 - \frac{1}{10}e^2 - \frac{8}{175}e^4.....\right]\left[1 + \frac{3}{8}\left\{1 - \frac{2}{5}e^2 - \frac{17}{175}e^4.....\right\}R + O\left(R^2\right)\right]$$

$$= 1 - \frac{1}{10}e^2 - \frac{8}{175}e^4 + \frac{3}{8}\left\{1 + \frac{3}{10}e^2 - \frac{18}{175}e^4.....\right\}R + O\left(R^2\right) \tag{19}$$

where $R = \left(\frac{\rho U a}{\mu}\right)$ is particle Reynolds number. The same solution may be re-written, when we take particle Reynolds number $R = \left(\frac{\rho U b}{\mu}\right)$, by using $b/a = (1-e^2)^{1/2}$, as

$$\frac{F}{F_y} = K\frac{F}{F_x} = K\left[1 + \frac{e^3}{\sqrt{1-e^2}\left[-2e + \left(1+e^2\right)\ln\frac{1+e}{1-e}\right]}R + O\left(R^2\right)\right],$$

$$= \left[1 - \frac{1}{10}e^2 - \frac{8}{175}e^4.....\right]\left[1 + \frac{e^3}{\sqrt{1-e^2}\left[-2e + \left(1+e^2\right)\ln\frac{1+e}{1-e}\right]}R + O\left(R^2\right)\right]$$

$$= \left[1 - \frac{1}{10}e^2 - \frac{8}{175}e^4.....\right]\left[1 + \frac{3}{8}\left\{1 + \frac{1}{10}e^2 + \frac{109}{1400}e^4.....\right\}R + O\left(R^2\right)\right]$$

$$= 1 - \frac{1}{10}e^2 - \frac{8}{175}e^4 + \frac{3}{8}\left\{1 + \frac{31}{1400}e^4\right\}R + O\left(R^2\right). \tag{20}$$

Equations (19) and (20) immediately reduces to the case of sphere(given in eq. 16) in the limiting case as $e \to 0$. On the other hand, the expressions (19) and (20) due to Oseen for prolate spheroid appears to be new as no such type of expressions are available in the literature for comparison.

Flow past oblate spheroid

Stokes drag on oblate spheroid having semi-major axis length 'b' and semi-minor axis length 'a' placed in uniform velocity U, parallel to axis of symmetry(axial flow) and perpendicular to its axis of symmetry(transverse flow) is given as[by utilizing DS conjecture 5 and 6, Datta and Srivastava, 1999]

$$F_x = 8\pi\mu U a e^3\left[e\sqrt{1-e^2}-\left(1-2e^2\right)\sin^{-1}e\right]^{-1}, \quad (21a)$$

$$F_y = 16\pi\mu U a e^3\left[-e\sqrt{1-e^2}+\left(1+2e^2\right)\sin^{-1}e\right]^{-1}. \quad (21b)$$

By using (21a,b) and (8), the value of real factor 'K' comes out to be

$$K = \frac{F_x}{F_y} = \frac{1}{2}\left[-e\sqrt{1-e^2}+\left(1+2e^2\right)\sin^{-1}e\right]\left[e\sqrt{1-e^2}-\left(1-2e^2\right)\sin^{-1}e\right]^{-1}$$

$$= 1-\frac{7}{30}e^2-\frac{199}{33600}e^4 \ldots\ldots \quad (22)$$

Now, from Brenner's formula (9) and (10), the Oseen's correction, with the use of real factor K(eq. 22), may be written as

$$\frac{F}{F_y} = K\frac{F}{F_x}$$

$$= \left[1-\frac{7}{30}e^2-\frac{199}{33600}e^4 \ldots\ldots\right]\left[1+\frac{3}{8}\left\{1-\frac{1}{10}e^2-\frac{31}{1400}e^4 \ldots\ldots\right\}R+O\left(R^2\right)\right]$$

$$= 1-\frac{7}{30}e^2-\frac{199}{33600}e^4+\frac{3}{8}\left\{1-\frac{1}{3}e^2-\frac{53}{11200}e^4 \ldots\ldots\right\}R+O\left(R^2\right) \quad (23)$$

where $R = \left(\frac{\rho U a}{\mu}\right)$ is particle Reynolds number. The same solution may be re-written, when we take particle Reynolds number $R = \left(\frac{\rho U b}{\mu}\right)$, by using $b/a = (1-e^2)^{1/2}$, as

$$\frac{F}{F_y} = K\frac{F}{F_x} = K\left[1+\frac{e^3}{2\sqrt{1-e^2}\left[e\sqrt{1-e^2}-\left(1-2e^2\right)\sin^{-1}e\right]}R+O\left(R^2\right)\right],$$

$$= \left[1-\frac{7}{30}e^2-\frac{199}{33600}e^4 \ldots\ldots\right]\left[1+\frac{e^3}{2\sqrt{1-e^2}\left[e\sqrt{1-e^2}-\left(1-2e^2\right)\sin^{-1}e\right]}R+O\left(R^2\right)\right]$$

$$= \left[1-\frac{7}{30}e^2-\frac{199}{33600}e^4 \ldots\ldots\right]\left[1+\frac{3}{8}\left\{1+\frac{2}{5}e^2+\frac{61}{200}e^4 \ldots\ldots\right\}R+O\left(R^2\right)\right]$$

$$= 1-\frac{7}{30}e^2-\frac{199}{33600}e^4+\frac{3}{8}\left\{1-\frac{1}{3}e^2-\frac{4079}{33600}e^4 \ldots\ldots\right\}R+O\left(R^2\right). \quad (24)$$

Equations (23) and (24) immediately reduces to the case of sphere(given in eq. 16) in the limiting case as $e \rightarrow 0$. On the other hand, the expressions (23) and (24) due to Oseen for oblate spheroid appears to be new as no such type of expressions are available in the literature for comparison.

CONCLUSION

General expressions of Oseen's correction to Stokes drag on axially symmetric particle placed in uniform stream parallel to axis of symmetry(longitudinal flow) and perpendicular to axis of symmetry(transverse flow) are derived with the help of DS-conjecture (Datta and Srivastava, 1999). These results are applied to sphere and spheroids to evaluate the Oseen's drag corrected up to first order terms in Reynolds number Re. It is interesting to note that these results work well to evaluate the important physical quantity like drag experienced by body without knowing the complete velocity field and stream function valid around the body placed in axial and transverse uniform stream. According to intensive literature survey presented in Introduction, it was found that **Brenner**(1961) gave a formula for writing Oseen's correction to Stokes drag on axially symmetric particle placed in axial flow. In the present problem, Brenner's formula is extended further for transverse flow. Author further claims here that the results presented in this paper may be explored for variety of complex flows with oblique angle of incidence, parabolic flow, hyperbolic flow etc. These results are helpful for calculating the settling of solid particles of powdered materials with enhanced accuracy in carrying out sedimentation analysis and in acoustic levitation. These two works are the main practical applications of Oseen's correction which was not possible with the Stokes drag.

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