International Journal of Current Advanced Research
ISSN: O: 2319-6475, ISSN: P: 2319-6505, Impact Factor: 6.614
Available Online at www.journalijcar.org
Volume 9; Issue 09(A); September 2020; Page No. 23038-23042
DOI: http://dx.doi.org/10.24327/ijcar.2020.23042.4556

Research Article

# A GEOMETRIC PROOF OF INFINITE SET HOMOGENEITY

**Anthony C. Patton**

San Diego, California, USA,

**ARTICLE INFO**

***Article History:***
Received 15th June, 2020
Received in revised form 7th July, 2020
Accepted 13th August, 2020
Published online 28th September, 2020

***Key words:***
infinity, potential infinity, cardinality, Cantor, Aristotle, one-to-one correspondence, continuum, numbers, Axiom of Infinity, Axiom of Power Set, infinitesimal, derivative, Hilbert's Hotel, Diagonal Argument.

**ABSTRACT**

This paper offers a geometric proof to demonstrate that all infinite sets have the same cardinality in relation to absolute infinity (more correctly, that all infinite sets are only potentially infinite), thus eliminating the need for Cantor's transfinite numbers, and analyzes the structure of our numbering system in perpetuating certain misunderstandings about infinite sets.



## INTRODUCTION

Since Aristotle, philosophers and mathematicians have pondered the distinction between infinity by addition and infinity by division.[1] Infinity by addition (1 + 1 + 1 + …) could not apply to particles in the universe because if the universe consisted of an infinite set of particles, each with magnitude, the universe would be infinitely large. The problem resolves if we accept a finite universe or consider numbers as abstract ideas or objects without magnitude, which generates the set of natural numbers.

Infinity by division (1/2 + 1/4 + 1/8 + …) could apply to particles in the universe because we could divide particles ad infinitum without creating the problem of an infinitely large universe. The methodology is problematic, however, if we consider that particles probably are not infinitely divisible. The problem resolves if we consider numbers as abstract ideas or objects without magnitude, which generates the set of rational numbers.

### *Decimal Symmetry*

The distinction between infinity by addition and infinity by division boils down to adifference between numbers on the left side of the decimal and numbers on the right side of the decimal.

For example, numbers like 1/3 and√2 each have a so-called countably infinite set of natural numbers on the right side of the decimal, which can be thought of as a complete set according to the Axiom of Infinity. If we cannot express √2 as an infinite decimal, then we cannot express 1/3 as an infinite decimal, which means 1/3 ≠ 0.333…. This point is often ignored.

For numbers on the left side of the decimal, however, the same logic apparently does not apply. If we propose that the natural numbers area complete infinite set, per the Axiom of Infinity, then the largest natural number would be represented as a numeral with a so-called countably infinite set of 9s on the left side of the decimal (…999,999.0). If such a number does not exist, then the Axiom of Infinity is invalid and the natural numbers are only potentially infinite. Cantor admits as much, which raises questions about the need for transfinite numbers.

Of interest, if infinite natural numbers are allowed, we can establish a one-to-one correspondence between the natural numbers and the real numbers by pairing up a mirror image of the two sets. The reordering of the set of real numbers based on their mirror image weight (0.1 < 0.01 < 0.001, etc.) avoids the density problem (no numbers exist between any two sequential numbers) and results in a list of the real numbers, which invalidates the diagonal argument.[2] If this infinite

**Corresponding author:* **Anthony C. Patton**
San Diego, California, USA,

[2] Another way to discredit the diagonal argument is to note that the same principle applies to an infinite list of rational numbers expressed as decimals.

process does not bridge the gap between the rational numbers and the real numbers, the Axiom of Infinity is invalid.

| Natural | Real[3] |
|---|---|
| 1.0 | 0.1 |
| 2.0 | 0.2 |
| 3.0 | 0.3 |
| … | … |
| …259141.0 | 0.141952… |
| … | … |
| …312414.0 | 0.414213… |
| … | … |
| …999,997.0 | 0.799,999… |
| …999,998.0 | 0.899,999… |
| …999,999.0 | 0.999,999… |

Before continuing, my claim about the Axiom of Infinity raises the important distinction between absolute infinity and transfinite infinity. A key assertion of this paperis that absolute infinity is the only valid concept for actual infinity because the concept of transfinite infinity does not properly distinguish between potential infinity and actual infinity, as Cantor himself admitted (emphasis added):

I wish to make a sharp contrast between the Absolute and what I call the Transfinite, that is, the actual infinities of the last two sorts [*in abstracto* and *in concreto*], which are *clearly limited, subject to further increase, and thus related to the finite*.[4]

In many ways, the ongoing debate about infinity boils down to this distinction. Thus, when I challenge Cantor's theory, I am challenging the foundation of his theory, not the results of his theory if we accept his foundation. My use of absolute infinity in this paper is designed to demonstrate the limits and contradictions that result from the idea of transfinite infinity.

Because infinite decimals are allowed (1/3 = 0.333…) and infinite natural numbers are not allowed, this gives the illusion that the natural numbers and the real numbers are not the same size. That is, based on a quirk or limitation of our base-10 numbering system, or a failure of our imagination or "freedom," this lack of decimal symmetry accounts for the apparent difference in size for the natural numbers and the real numbers.

### *Geometrical Numbers*

Historically, the relationship between numbers and geometry has played an important role in our understanding of mathematics. One of the best ways to understand mathematics is to consider numbers represented asgeometry (e.g., functions represented on a Cartesian plane) or geometry represented as numbers (e.g., parabolic falls represented as functions). The Greeks' concerns about irrational numbers shifted their emphasis from numbers to geometry (with compass and ruler on a Euclidean plane). Modern mathematics shifted from geometry (with Cartesian planes) to numbers and sets with rigorous definitions. We now see a shift back to geometry, with a focus on symmetry.

We can use geometry to help us better understand numbers and we can use numbers to help us better understand geometry. For example, Descartes invented analytic geometry and the Cartesian plane to visually represent functions, which, among other things, allows students to see where derivatives equal 0. One epistemic downside of this powerful tool is the tendency to view the two domains (numbers and geometry) as identical, rather than as distinct but capable of representing the other. The Cartesian plane give us a visual display of the continuum, which prompts our minds to conclude that the numerical continuum must therefore exist and that our numbering system must therefore be continuous (the real numbers) to establish a one-to-one correspondence between the two domains.

### Geometric Cardinality

The geometric continuum is intuitive, not rigorous, and helps us resolve the problem of "gaps," even though science rarely arrives at even twenty decimal points of accuracy. The most important point for this paper is to use geometry to demonstrate that the natural numbers, the rational numbers, and the real numbers have the same cardinality—i.e., to establish a one-to-one correspondence among them. Traditionally, the p/q definition of the rational numbers is used to prove the existence of "gaps" along the continuum: the division of two (finite) integers never generates an irrational number. This supposedly proves that the real numbers are larger than the rational numbers, but this problem resolves if we allow for infinite numbers or infinite fractions.[5]

Consider a circle with arbitrary circumference 1.[6] If we arbitrarily designate one point as 0 and rotate the circle, we can note the number [0, 1] at the point where it stops. Because the circumference is a continuum, the assumption is that the real numbers must define the infinite set of points along the circumference. In other words, a one-to-one correspondence is purported to exist between the set of real numbers and the infinite set of points along the circumference of the circle. If this is true, than any other sets of numbers that can generate the same circle would have the same cardinality as the real numbers.

Consider the rational numbers [0, 1]. If we divide the circle into three equal portions, we have points at 0, 1/3, and 2/3; and if we draw lines connecting these points we have a triangle. If we divide the circle into four equal portions, we have points at 0, 1/4, 2/4, and 3/4; and if we draw lines connecting these four points we have a square. If we divide the circle into five equal portions, we have points at 0, 1/5, 2/5, 3/5, and 4/5; and if we draw lines connecting these five points we have a pentagon, and so on. Because the circle has a circumference of 1, the

[3] This list includes every finite and infinite decimal reordered as mirror images of every finite and infinite natural number, to include 1/3, $\pi - 3$, and $\sqrt{2} - 1$.

[4] Georg Cantor, *Gesammelte Abhandlungen*, p. 378. Quoted in Rudy Rucker (1982), p. 9.

[5] We can also resolve the problem of "gaps" with rational numbers if we define the rational numbers as decimals and allow them to go to infinity, like repeating decimals.

[6] The unit of measurement for every continuum is always arbitrary, is therefore not grounded in Aristotle's category of quantity, and is therefore not objectively or metaphysically definable as "1."

length of each circumference interval equals the inverse of the quantity of segments:

$$1/3, 1/4, 1/5, \dots , 1/n$$

The limit of 1/n as n goes to infinity is 0, not an infinitesimal. Calculus eliminates infinitesimals and defines the limit as 0. In the case of a circle, therefore, as 1/n goes to infinity, the regular polygons transform until the lengths of the sides (the intervals between the points) goes precisely 0. That is, the limit results in an infinite set of dimensionless points. A regular polygon with a so-called countably infinite number of sides is an apeirogon.

Given that an apeirogon is nothing more than this infinite set of points equidistance from a central point, the definitions of apeirogon and circle area distinction without a difference. To reject this is to reject Calculus—the limit is 0. Given that a circle therefore represents both the rational numbers and the real numbers, both sets must have the same cardinality.

Now consider the natural numbers. By the same logic, we can allow the natural numbers to represent regular polygons—3 = triangle, 4 = square, 5 = pentagon, … , 1,000 = chiliagon, and so on. In this case, however, the equidistant points along the circumference represent a set of natural numbers not defined by their position (infinity by addition), not a set of rational numbers defined by their position (infinity by division). As the limit of 1/n goes to infinity, the regular polygons transform until the length of the sides (the intervals between the points) goes 0 and generates a circle. This means the real numbers, the rational numbers, and the natural numbers all have the same cardinality.

This proof demonstrates the false distinction between numbers on the left and right sides of the decimal: adding equidistant points along a circumference (infinity by addition with the natural numbers) is the same as dividing a circumference with equidistant points (infinity by division with the rational numbers). If Calculus is valid and the limit of 1/n as n goes to infinity is 0, then this allows us to bridge the gap between the rational numbers and the real numbers.

***Power Set Methodology Error***

One of the biggest challengers with the real numbers as currently defined is that there is no methodology to generate the set from a finite starting point, unless we allow for infinite numbers or infinite fractions. This means there is no way to generate a circle with the real numbers. Instead, a circle generates the real numbers for us to discover, which seems at odds with the idea of mathematics as being grounded in a finite set of principles or axioms.

Beginning with the number 1 and the operation of addition, we can generate the sets of natural numbers, integers, and rational numbers. With the inclusion of the exponent operation, we can generate the algebraic irrationals and rational complex numbers. From this, it follows that sets are so-called countably infinite if they can be generated ad infinitum from the finite starting point of 1 and addition. However, there is no similar methodology for generating the set of so-called transcendental irrationals,[7] which are presumed to exist nonetheless. This is the basis for saying the real numbers are larger than the natural numbers or that the real numbers fill the “gaps” on the continuum left by the rational numbers.

Even though there is no methodology for generating the set of real numbers, Cantor claimed to know their relativesize: if the so-called countably infinite sets have cardinality $\infty$, then the real numbers have cardinality $2^{\infty}$. Cantor used the power set to explain the difference, noting that for any set with n elements, the set of all subsets (the power set) has $2^n$ elements.[8] For example, for the set {1, 2, 3}, the set of all subsets includes {0}, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, and {1, 2, 3}, which means that if a set contains 3 elements, the power set contains $2^3$ or 8 elements, which is larger, even, we are told, for infinite sets.

Cantor claimed he could establish a one-to-one correspondence between the real numbers and the power set of the natural numbers, represented as infinite binary decimals, even though the real numbers are base-10 infinite decimals.[9] This is why if the set of natural numbers has cardinality $\infty$, the set of real numbers has cardinality $2^{\infty}$. It is important to note that Cantor did not generate the set of real numbers from the natural numbers. Rather, he generated an infinite set (the power set of the natural numbers) with cardinality supposedly equal to the cardinality of the real numbers, which supposedly allowed him to establish a one-to-one correspondence between the two sets.

The important point for our analysis is to demonstrate a problem with the power set methodology. Returning to the methodology for generating a circle with the natural numbers or the rational numbers, what if, rather than add one regular polygon side at a time, we double the number of sides each time? Both methodologies rely on an infinite series of natural numbers, but at each step of the way the cardinality of the second methodology equals the power set ($2^n$) of the first methodology.

| $n$ | $2^n$ |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 8 |
| 4 | 16 |
| 5 | 32 |

If the first methodology with cardinality $\infty$ generates a circle, what does the second methodology with cardinality $2^{\infty}$generate? The geometry of a circle does not allow for different results because the set of natural numbers is sufficient to generate a circle (no “gaps”). If so, then the power set of the natural numbers is not larger than the natural numbers, which discredits the idea of a hierarchy of infinite sets and suggests

[7] The mirror image methodology above lists them but does not generate them.

[8] Given that no proofs exist for infinite sets, they are presumed to exist in set theory with the Axiom of Infinity and the Axiom of Power Set.

[9] Equating the sets of infinite binary decimals and base-10 infinite decimals is possibly only if the two sets are implicitly understood as potentially infinite.

that all infinite sets are all the same size or only potentially infinite. Not to mention, how does one take the power set of a set that never ends? Either it does not make sense to take the power set of an infinite set, because sets are only potentially infinite, or doing so does not make the infinite set larger, because the set of natural numbers is sufficient to generate a circle.

## CONCLUSION

The best way to discredit an argument is to discredit its assumptions. For example, if one side argues that a one-to-one correspondence exists between numbers and geometry, such as the real numbers and the circumference of a circle, then if we can generate the same circle using other infinite sets that are supposed to be smaller, such as the natural numbers or the rational numbers, then the three sets necessarily have the same cardinality. This means set theory implicitly considers the natural numbers and the rational numbers as only potentially infinite.

The problem lies in the assumptions we make about infinite numbers and the Axiom of Infinity. Just as analytic philosophy and modern logic revealed some problems with our ordinary language, a review of the base-10 numbering system is in order.[10] If we invoke Cantor's own words, "The essence of mathematics is freedom," then a full implementation of his call to freedom would discredit his own conclusions. In short, either all infinite sets have the same cardinality or all infinite sets are only potentially infinite.

## References


Alexander, Amir, *Infinitesimal: How a Dangerous Mathematical Theory Shaped the Modern World* (New York: Farrar, Straus and Giroux, 2014).

Aristotle, *The Basic Works of Aristotle*, edited by Richard McKeon (New York: Random House, 1941).

Bagaria, Joan, "Set Theory", *The Stanford Encyclopedia of Philosophy* (Winter 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2014/entries/set-theory/>.

Balaguer, Mark, "Fictionalism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Summer 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2015/entries/fictionalism-mathematics/>.

Bell, John L., "Continuity and Infinitesimals", *The Stanford Encyclopedia of Philosophy* (Winter 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2014/entries/continuity/>.

Bridges, Douglas and Palmgren, Erik, "Constructive Mathematics", *The Stanford Encyclopedia of Philosophy* (Winter 2013 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2013/entries/mathematics-constructive/>.

Bueno, Otávio, "Nominalism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Spring 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2014/entries/nominalism-mathematics/>.

Colyvan, Mark, "Indispensability Arguments in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Spring 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2015/entries/mathphil-indis/>.

Dantzig, Tobias, *Number: The Language of Science* (New York: Plume, 2007).

Dauben, Joseph Warren, "Georg Cantor: His Mathematics and Philosophy of the Infinite" (Cambridge: Harvard University Press, 1979).

Ellenberg, Jordan, *How Not to Be Wrong: The Power of Mathematical Thinking* (New York: Penguin, 2014).

Franklin, James, *An Aristotelian Realist Philosophy of Mathematics: Mathematics as the Science of Quantity and Structure* (New York: Palgrave Macmillin, 2014).

Frenkel, Edward, *Love & Math: The Heart of Hidden Reality* (New York: Basic Books, 2013).

George, Alexander; Velleman, Daniel J., *Philosophies of Mathematics* (Malden, MA: Blackwell, 2002).

Havil, Julian, *The Irrationals: A Story of the Numbers You Can't Count On* (Princeton: Princeton University Press, 2012).

Iemhoff, Rosalie, "Intuitionism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Spring 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/spr2015/entries/intuitionism/>.

Linnebo, Øystein, "Platonism in the Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Winter 2013 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2013/entries/platonism-mathematics/>.

Mancosu, Paolo, "Explanation in Mathematics", *The Stanford Encyclopedia of Philosophy* (Summer 2015 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2015/entries/mathematics-explanation/>.

Mendell, Henry, "Aristotle and Mathematics", *The Stanford Encyclopedia of Philosophy* (Winter 2008 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/win2008/entries/aristotle-mathematics/>.

Merzbach, Uta C.; Boyer, Carl B., *A History of Mathematics*, third edition (Hoboken, NJ: John Wiley & Sons, 2011).

Núñez, Rafael E., "Creating mathematical infinities: Metaphor, blending, and the beauty of transfinite cardinals" (Journal of Pragmatics, 37 (2005), 1717 – 1741).

Paseau, Alexander, "Naturalism in the Philosophy of


[10] The problem of 0.999… resolves in a base-9 numbering system.

Mathematics", *The Stanford Encyclopedia of Philosophy* (Summer 2013 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2013/entries/naturalism-mathematics/>.

Patton, Anthony C. *The End of Infinity: Where Mathematics and Philosophy Meet*, (New York: Algora Publishing, 2018).

—The Mathematical Errors of Infinity (2019).*Online Mathematics,* 2(1). <www.onlinemathematics.net/Articles/V2S1/OMJV2I1 Antonv3.pdf>.

Reichenbach, Bruce, "Cosmological Argument", *The Stanford Encyclopedia of Philosophy* (Winter 2016 Edition), Edward N. Zalta (ed.), URL = <https://plato.stanford.edu/archives/win2016/entries/cosmological-argument/>.

Rodych, Victor, "Wittgenstein's Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Summer 2011 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/sum2011/entries/wittgenstein-mathematics/>.

Russell, Bertrand, *Introduction to Mathematical Philosophy* (Digireads.com Publishing, 2010).

Schopenhauer, Arthur, *On the Fourfold Root of the Principle of Sufficient Reason* (La Salle, IL: Open Court, 1999).

Seife, Charles, *Zero: The Biography of a Dangerous Idea* (New York: Penguin, 2000).

Shabel, Lisa, "Kant's Philosophy of Mathematics", *The Stanford Encyclopedia of Philosophy* (Fall 2014 Edition), Edward N. Zalta (ed.), URL = <http://plato.stanford.edu/archives/fall2014/entries/kant-mathematics/>.

Shapiro, Steward, *Thinking About Mathematics: The Philosophy of Mathematics* (New York: Oxford University Press, 2011).

Struik, Dirk J., *A Concise History of Mathematics* (New York: Dover, 1987).

Zenkin, Alexander A., "Logic of Actual Infinity and G. Cantor's Diagonal Proof of the Uncountability of the Continuum," *The Review of Modern Logic* (Volume 9, Numbers 3 & 4, December 2003 – August 2004, Issue 30).

**How to cite this article:**

Anthony C. Patton (2020) 'A Geometric Proof of Infinite Set Homogeneity', *International Journal of Current Advanced Research,* 09(09), pp. 23038-23042. DOI: http://dx.doi.org/10.24327/ijcar.2020. 23042.4556

*******